MATH 236 CALCULUS 2
STRATEGY FOR TESTING SERIES

1. Geometric series: $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$ if and only if $-1 < r < 1$.
2. p-series: $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges if and only if $p > 1$.
3. If $\lim_{n\to\infty} a_n \neq 0.$, then $\sum_{n=1}^{\infty} a_n$ diverges.
4.* Integral Test : $\{a_n\}$ must be positive and decreasing.
5.* Direct Comparison Test
6.* Limit Comparison Test : $\lim_{n\to\infty} \dfrac{a_n}{b_n} = number > 0$.
7.* Ratio Test : $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = number$. Compare this number with one.
8.* Root Test : $\lim_{n\to\infty} \sqrt[n]{a_n} = number$. Compare this number with one.
9. Alternating Series Test.
10. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

$*$ means that all terms in the series must be non-negative.

Determine whether the series is convergent or divergent.

1. $\sum_{n=1}^{\infty} \dfrac{n+1}{2n-3}$ $\lim_{n\to\infty} \frac{n+1}{2n-3} = \frac{1}{2} \neq 0$, so the series diverges.

2. $\sum_{n=1}^{\infty} \dfrac{\sqrt{n^3+1}}{3n^3+4n^2+1}$ $0 < \frac{\sqrt{n^3+1}}{3n^3+4n^2+1} < \frac{\sqrt{n^3+1}}{3n^3} \le \frac{\sqrt{2n^3}}{3n^3} = \frac{\sqrt{2}\cdot n^{\frac{3}{2}}}{3\cdot n^3} = \frac{\sqrt{2}}{3}\cdot\frac{1}{n^{\frac{3}{2}}}$.

Convergent by the direct comparison test.

$\sum_{n=1}^{\infty} \frac{\sqrt{2}}{3}\cdot\frac{1}{n^{3/2}} = \frac{\sqrt{2}}{3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ convergent p-series with $p=\frac{3}{2}$. convergent

3. $\sum_{n=1}^{\infty} \dfrac{1}{4^n-1}$ Convergent by the Limit Comparison Test.

$\lim_{n\to\infty} \dfrac{\frac{1}{4^n-1}}{\frac{1}{4^n}} = \lim_{n\to\infty} \dfrac{4^n}{4^n-1} = \lim_{n\to\infty} \dfrac{1}{1-\frac{1}{4^n}} = 1 > 0$

$\sum_{n=1}^{\infty} \frac{1}{4^n}$ is a convergent geometric series.

4. $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^3}{n^4+1}$ Convergent by the Alternating Series Test.

i) $\lim_{n\to\infty} \frac{n^3}{n^4+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^4}} = \frac{0}{1} = 0$.

ii) $f(x) = \frac{x^3}{x^4+1}$, $f'(x) = \frac{3x^2(x^4+1) - x^3\cdot 4x^3}{(x^4+1)^2} = \frac{-x^6+3x^2}{(x^4+1)^2} < 0$ for all $x \geq 2$.

5. $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$

$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \dfrac{\cancel{n!}\,\cancel{2^{n+1}}\,2}{\underbrace{\cancel{(n+1)!}}_{n+1}\,\cancel{2^n}} = \lim_{n\to\infty} \dfrac{2}{n+1} = 0 < 1$

Convergent by the Ratio Test.

6. $\sum_{n=1}^{\infty} (-1)^{n+1} n^2 (\frac{2}{3})^n$ ← Show absolute convergence, i.e., $\sum_{n=1}^{\infty} n^2(\frac{2}{3})^n$ converges by using the Root test:

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{n^2(\frac{2}{3})^n} = \lim_{n\to\infty} n^{\frac{2}{n}}\cdot\frac{2}{3} = 1\cdot\frac{2}{3} < 1$

※ $\lim_{n\to\infty} n^{\frac{2}{n}} = \lim_{n\to\infty} e^{\ln n^{\frac{2}{n}}} = e^{\lim_{n\to\infty}\frac{2\ln n}{n}} \underset{L.R.}{=} e^0 = 1$

7. $\sum_{n=1}^{\infty} 2ne^{-n^2}$ convergent → $a_n = \frac{2n}{e^{n^2}}$ is positive and decreasing. ✓

$\int_1^{\infty} 2x\cdot e^{-x^2}\,dx$

$= \lim_{t\to\infty} \int_1^t 2x\cdot e^{-x^2}\,dx$ $u = -x^2$, $du = -2x\,dx$, $\int_1^t \to \int_{-1}^{-t^2}$

$= \lim_{t\to\infty} -\int_{-1}^{-t^2} e^u\,du$

$= \lim_{t\to\infty} [-e^{-t^2} + e^{-1}]$

$= \lim_{t\to\infty} [-\frac{1}{e^{t^2}} + \frac{1}{e}] = -0 + \frac{1}{e} = \boxed{\frac{1}{e}}$ #

7 STRATEGY FOR TESTING SERIES (Reference : J. Stewart)

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form*.

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\Sigma\, (-1)^{n-1}b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).