MATH235 Calculus 1
Homework problems on the Squeeze Theorem.
09/08/2010

Use the Squeeze Theorem to solve the following problems:

1. Find $\lim_{x\to 0} x^4 \cos(\frac{2}{x})$.
Since the range of cosine is always between -1 and 1,

$$-1 \leq \cos(\frac{2}{x}) \leq 1.$$

By multiplying $x^4 \geq 0$ to each term in this inequality, we get

$$-x^4 \leq x^4 \cos(\frac{2}{x}) \leq x^4.$$

Therefore, since

$$\lim_{x\to 0} -x^4 = 0 = \lim_{x\to 0} x^4,$$

by the Squeeze Theorem, we have that $\lim_{x\to 0} x^4 \cos(\frac{2}{x}) = 0$.

2. Find $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})}$.
Since the range of sine is always between -1 and 1,

$$-1 \leq \sin(\frac{1}{x}) \leq 1.$$

Now, since $y = e^x$ is an increasing function, this inequality implies that

$$e^{-1} \leq e^{\sin(\frac{1}{x})} \leq e^1,$$

and by multiplying $x^2 \geq 0$ to each term above, we get

$$x^2 e^{-1} \leq x^2 e^{\sin(\frac{1}{x})} \leq x^2 e^1.$$

Hence, since

$$\lim_{x\to 0} x^2 e^{-1} = 0 = \lim_{x\to 0} x^2 e^1,$$

by the Squeeze Theorem, it follows that $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})} = 0$.

3. Let f be a function that satisfies $|f(x)| \leq M$ for all $x \neq 0$. Show that $\lim_{x\to 0} xf(x) = 0$. Hint : $|xf(x)| = |x||f(x)|$.
Since $|f(x)| \leq M$, we have

$$|xf(x)| = |x||f(x)| \leq |x|M.$$

In other words,

$$-|x|M \leq xf(x) \leq |x|M.$$

Therefore, since

$$\lim_{x\to 0} -|x|M = 0 = \lim_{x\to 0} |x|M,$$

by the Squeeze Theorem, we conclude that

$$\lim_{x\to 0} xf(x) = 0.$$