MATH235 Calculus 1
Proof of the Squeeze Theorem.

Theorem 0.1 (The Squeeze Theorem). *Suppose that* $g(x) \leq f(x) \leq h(x)$ *for all* x *in some open interval containing* c *except possibly at* c *itself. If* $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$ *then* $\lim_{x\to c} f(x) = L$.

Proof. Let $\epsilon > 0$ be given. We are done if we find a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } 0 < |x - c| < \delta.$$

Since $\lim_{x\to c} g(x) = L$, by definition of limits, there exists some $\delta_1 > 0$ such that

$$|g(x) - L| < \epsilon \text{ for all } 0 < |x - c| < \delta_1.$$

Thus,

$$-\epsilon < g(x) - L < \epsilon \text{ for all } 0 < |x - c| < \delta_1,$$

so

$$L - \epsilon < g(x) < L + \epsilon \text{ for all } 0 < |x - c| < \delta_1. \tag{1}$$

Similarly, since $\lim_{x\to c} h(x) = L$, by definition of limits, there exists some $\delta_2 > 0$ such that

$$L - \epsilon < h(x) < L + \epsilon \text{ for all } 0 < |x - c| < \delta_2. \tag{2}$$

Additionally, since $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, there exists some $\delta_3 > 0$ such that

$$g(x) \leq f(x) \leq h(x) \text{ for all } 0 < |x - c| < \delta_3. \tag{3}$$

Now, we choose $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then by (1), (3), and (2), we have that

$$L - \epsilon < g(x) \leq f(x) \leq h(x) < L + \epsilon \text{ for all } 0 < |x - c| < \delta.$$

Therefore, $-\epsilon < f(x) - L < \epsilon$ for all $0 < |x - c| < \delta$, so

$$|f(x) - L| < \epsilon \text{ for all } 0 < |x - c| < \delta.$$

Hence, by definition of limits, $\lim_{x\to c} f(x) = L$. □