Worksheet 4 MATH 235 Name: ____________________

Fall, 2010.

Discuss the following problems with your group and write down a complete solution. Show all work.

1. Prove that if f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof. To show that f is continuous at $x = c$, we show that $f(c)$ and $\lim_{x\to c} f(x)$ both exists and that $\lim_{x\to c} f(x) = f(c)$. First of all, it is clear that the function is defined at $x = c$, since it is given that f is not only defined but also differentiable at $x = c$. We only need to check that $\lim_{x\to c} f(x)$ does exist and that this value is indeed the same as $f(c)$. Note that, by defining $h = x - c$, we have that

$$\lim_{x\to c} f(x) = \lim_{h\to 0} f(c+h). \tag{0.1}$$

Additionally, we have that

$$f(c+h) = f(c+h) - f(c) + f(c) = \frac{f(c+h) - f(c)}{h} \cdot h + f(c). \tag{0.2}$$

Therefore,

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{h\to 0} f(c+h) && \text{by (0.1),}\\
&= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot h + f(c) && \text{by (0.2),}\\
&= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \lim_{h\to 0} h + \lim_{h\to 0} f(c) && \text{since } f(x) \text{ is differentiable at } x = c,\\
&= f'(c)\cdot 0 + f(c) = f(c).
\end{aligned}$$

Notice that if $\lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h}$ did not exists, we would not be able to split the limit as in the third equality, since we could split limits only when each limit appearing after the split exists as a number. We do know, however, that $\lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h}$ exists, since it is given that f is differentiable at $x = c$.

This shows that $\lim_{x\to c} f(x)$ exists, and it is equal to $f(c)$. Therefore, f is continuous at $x = c$, and the proof is complete. □

2. Is the converse of the previous statement true? In other words, is it true that if f is continuous at $x = c$, then f is differentiable at $x = c$? We can show that this statement is FALSE by finding a counterexample. Show that $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$.

Counter example: We show that $f(x) = |x|$ does not have a derivative at $x = 0$ although it is continuous at $x = 0$. First of all, $f(x) = |x|$ is continuous at $x = 0$, since

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

and

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0$$

so that $f(0) = 0 = \lim_{x\to 0} |x|$. $f(x) = |x|$ is not differentiable at $x = 0$, however, since $f'(0)$ does not exist. In particular, since

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

and

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

$f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist.