Worksheet 5 MATH 235 Name: ______________
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Discuss the following problems with your group and write down a complete solution. Show all work.

1. Prove the Product Rule: if f and g are differentiable, then so is their product $P(x) = f(x)g(x)$, and

$$P'(x) = f'(x)g(x) + f(x)g'(x).$$

- step 1. Write down the limit definition of $P'(x)$ in terms of f and g.

- step 2. Subtract $f(x+h)g(x)$ from the first term of the numerator and add $f(x+h)g(x)$ to the second term of the numerator. Convince yourself that you are simply rewriting your answer in step 1 by doing so without changing anything.

- step 3. Make manipulations in step 2 to reach:

$$\lim_{h\to 0}(f(x+h) \cdot \frac{g(x+h)-g(x)}{h} + g(x) \cdot \frac{f(x+h)-f(x)}{h}).$$

- step 4. We would like to split the limit in step 3 into separate limits of each term appearing there, but in order to do so, we know that the limits $\lim_{h\to 0} f(x+h)$, $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$, $\lim_{h\to 0} g(x)$, and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ must all exist. Verify that each of these limits do exist. Then, write down what each of those limits are, and conclude your argument. To verify that $\lim_{h\to 0} f(x+h)$ exists and it is in fact equal to $f(x)$, explain why f is continuous, and why continuity of f implies $\lim_{h\to 0} f(x+h) = f(x)$.

- By putting the above steps together, write down a complete mathematical proof.

2. Prove the Quotient Rule: if f and g are differentiable, then so is their quotient $Q(x) = \frac{f(x)}{g(x)}$, and

$$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

- step 1. Write down the limit definition of $Q'(x)$ in terms of f and g.

- step 2. Rewrite the answer in step 1 so that you do not see fractions in the numerator.

- step 3. Make manipulations in step 2 to reach:
  $$\lim_{h\to 0} \frac{1}{g(x+h)g(x)}(g(x) \cdot \frac{f(x+h) - f(x)}{h} - f(x) \cdot \frac{g(x+h) - g(x)}{h}).$$

- step 4. Verify, as in step 4 of the proof of the Product Rule, that we can split the limit in step 3 into separate limits of each term appearing there and conclude your argument.

- By putting the above steps together, write down a complete mathematical proof.