Worksheet 5 MATH 235
10/07/2010

Discuss the following problems with your group and write down a complete solution. Show all work.

1. Prove the Product Rule: if f and g are differentiable, then so is their product $P(x) = f(x)g(x)$, and

$$P'(x) = f'(x)g(x) + f(x)g'(x).$$

Proof. By the limit definition of derivatives and the definition of $P(x)$, we have

$$\begin{aligned} P'(x) &= \lim_{h\to 0} \frac{P(x+h) - P(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}. \end{aligned}$$

We can rewrite the last equality above, by adding and subtracting $f(x)g(x+h)$, as

$$\begin{aligned} P'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h}\right) \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \end{aligned} \tag{0.1}$$

We can split the limit as in the last equality, since each of the limit appearing there exists. In particular, since f and g are given to be differentiable, we know that

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= f'(x) \\ \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} &= g'(x) \end{aligned} \tag{0.2}$$

exist. Additionally, since differentiability implies continuity, we know that g is continuous so that

$$\lim_{z\to x} g(z) = g(x)$$

for all x in the domain of g by definition of a continuous function. Then, by rewriting the limit appearing above by using $h = z - x$, it follows that

$$\lim_{h\to 0} g(x+h) = g(x). \tag{0.3}$$

Lastly, it is clear that

$$\lim_{h\to 0} f(x) = f(x). \tag{0.4}$$

Therefore, by (0.2), (0.3), (0.4), and (0.1), we conclude that

$$P'(x) = f'(x)g(x) + f(x)g'(x),$$

and this completes the proof of the Product Rule. □

2. Prove the Quotient Rule: if f and g are differentiable, then so is their quotient $Q(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$, and

$$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Proof. By the limit definition of derivatives and the definition of $Q(x)$, we have

$$\begin{aligned} Q'(x) &= \lim_{h \to 0} \frac{Q(x+h) - Q(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}. \end{aligned}$$

Then, by simplifying and manipulating the given formulas, we reach

(0.5)

$$\begin{aligned} Q'(x) &= \lim_{h \to 0} \frac{\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\ &= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \\ &= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x) - f(x) \cdot \frac{g(x+h) - g(x)}{h} \right) \\ &= \frac{1}{\lim_{h \to 0} g(x+h) \lim_{h \to 0} g(x)} \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x) - \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right). \end{aligned}$$

We can split the limit as in the last equality above, since we know that each of the limit appearing there exists. In particular, since f and g are given to be differentiable, we know that

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= f'(x) \\ \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} &= g'(x) \end{aligned} \tag{0.6}$$

exist. Additionally, since g is differentiable, it is also continuous so that

$$\lim_{h \to 0} g(x+h) = g(x). \tag{0.7}$$

Lastly, it is clear that

$$\begin{aligned} \lim_{h \to 0} f(x) &= f(x) \\ \lim_{h \to 0} g(x) &= g(x). \end{aligned} \tag{0.8}$$

Therefore, by (0.6), (0.7), (0.8), and (0.5), we conclude that

$$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2},$$

and this completes the proof of the Quotient Rule. □