Worksheet 6 MATH 235
10/21/2010

1. (The Derivative Rule for Inverses.) Let f be a continuous one-to-one function defined on an interval. Suppose f is differentiable at $x = a$ and $f'(a) \neq 0$. If $f(a) = b$, show that $(f^{-1})'(b)$ exists and that

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Proof. By the limit definition of $(f^{-1})'(b)$, we have that

$$(f^{-1})'(b) = \lim_{h\to 0} \frac{f^{-1}(b+h) - f^{-1}(b)}{h}. \tag{0.1}$$

Define $k \in \mathbb{R}$ so that

$$f^{-1}(b+h) = a + k \qquad \text{or} \qquad f(a+k) = b + h. \tag{0.2}$$

Then, since $f^{-1}(b) = a$ and $f(a) = b$, by (0.2) and (0.1), we get

$$\begin{aligned} (f^{-1})'(b) &= \lim_{h\to 0} \frac{a + k - a}{h} \\ &= \lim_{h\to 0} \frac{k}{f(a+k) - f(a)}. \end{aligned} \tag{0.3}$$

Notice that, as h approaches zero, $k = f^{-1}(b+h) - a$ also approaches zero, since

$$\lim_{h\to 0}(f^{-1}(b+h) - a) = f^{-1}(\lim_{h\to 0}(b+h)) - a = f^{-1}(b) - a = 0,$$

where we have used the continuity of f^{-1} in the first equality above. Note that f^{-1} is continuous, since f is continuous. Therefore, we can rewrite (0.3) as

$$\begin{aligned} (f^{-1})'(b) &= \lim_{k\to 0} \frac{k}{f(a+k) - f(a)} \\ &= \frac{1}{f'(a)} && \text{since } \lim_{k\to 0} \frac{f(a+k) - f(a)}{k} = f'(a), \\ &= \frac{1}{f'(f^{-1}(b))}. \end{aligned}$$

$\square$