MATH 236 CALCULUS 2
WORKSHEET 7
MOMENTS AND CENTER OF MASS

1. Suppose $f(x) \geqslant g(x)$ on $[a, b]$ and consider the region between $y = f(x)$ and $y = g(x)$ on $[a, b]$. If this region has constant density δ, show that the center of mass (or centroid) of this region is located at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\delta \int_a^b x(f(x) - g(x))dx}{\delta \int_a^b f(x) - g(x)dx} \qquad \text{and} \qquad \bar{y} = \frac{\frac{1}{2}\delta \int_a^b f(x)^2 - g(x)^2 dx}{\delta \int_a^b f(x) - g(x)dx}.$$

ex) Find the centroid of the region bounded by $y = x^3$ and $y = \sqrt[3]{x}$.

ex) Find the centroid of the region under the curve $y = \sin x$, $0 \leqslant x \leqslant \pi$.

2. ($Pappus'$ Theorem) Prove that if a region R, lying on one side of a line in its plane, is revolved about that line, then the volume of the resulting solid is equal to the area of R multiplied by the distance traveled by its centroid.

- **Step 1.** Let R be the region between $y = f(x)$ and $y = g(x)$ on $[a, b]$, where $f(x) \geqslant g(x)$ and $0 \leqslant a \leqslant b$. Considering rotating this region R with respect to the y-axis. Draw this set up on the xy-plane.

- **Step 2.** Use the cylindrical shell method to find an integral that represents the volume of the resulting $3D$ object.

- **Step 3.**

Find the integral that represents the area of R. Then write down the distance traveled by the centroid of R by using the formula on previous page.

- **Step 4.**

Use step 2 and 3 to complete the proof.

3. By using $Pappus'$ Theorem, find the volume of the following objects.

(1) The torus made by rotating a circle with radius a units in a way that the distance between the center of this circle and the center of the torus is b units.

(2) Cone with radius r and height h.