① $\int 4x\sec^2(2x)\,dx$ $\qquad u = 4x \qquad dv = \sec^2(2x)\,dx$

$du = 4dx \qquad v = \frac{1}{2}\tan(2x)$

$= 4x\cdot\frac{1}{2}\tan(2x) - \int \frac{1}{2}\tan(2x)\cdot 4\,dx$

$= 2x\tan(2x) - 2\int \tan(2x)\,dx$

$\underset{\text{by ☆}}{=} 2x\tan(2x) - 2\left(-\frac{1}{2}\ln|\cos(2x)|\right) + C$

$= \boxed{2x\tan(2x) + \ln|\cos(2x)| + C}$

※ $\int \tan(2x)\,dx = \int \frac{\sin(2x)}{\cos(2x)}\,dx$ $\qquad u = \cos(2x)$

$du = -2\sin(2x)\,dx$

$= \int \frac{1}{u}\cdot\left(-\frac{1}{2}\right)du$ $\qquad -\frac{1}{2}du = \sin(2x)\,dx$

$= -\frac{1}{2}\int \frac{1}{u}\,du$

$= -\frac{1}{2}\ln|u| + C$

$= -\frac{1}{2}\ln|\cos(2x)| + C$ — ☆

※ In general,

$$\int \tan(ax)\,dx = -\frac{1}{a}\ln|\cos(ax)| + C$$

If $a=1$, then

$$\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C.$$

②

$$\int_1^e x^3 \ln x \, dx$$

$u = \ln x$ $\quad dv = x^3 dx$

$du = \frac{1}{x} dx$ $\quad v = \frac{1}{4} x^4$

$$= \left[\frac{1}{4}x^4 \ln x\right]_1^e - \int_1^e \frac{1}{4}x^4 \cdot \frac{1}{x} dx$$

$$= \left(\frac{1}{4}e^4 \ln e - \frac{1}{4} \cdot 1 \cdot \ln 1\right) - \int_1^e \frac{1}{4} x^3 dx$$

$$= \left(\frac{1}{4}e^4 - 0\right) - \frac{1}{4}\left[\frac{1}{4}x^4\right]_1^e$$

$$= \frac{1}{4}e^4 - \frac{1}{4}\left(\frac{1}{4}e^4 - \frac{1}{4}\right)$$

$$= \left(\frac{1}{4} - \frac{1}{16}\right)e^4 + \frac{1}{16} = \boxed{\frac{3}{16}e^4 + \frac{1}{16}}.$$

③

$$\int \sin^{-1} y \, dy = \int \underset{\substack{\uparrow \\ I}}{1} \cdot \underset{\substack{\uparrow \\ D}}{\sin^{-1} y} \, dy$$

$$= y \sin^{-1} y - \int y \frac{1}{\sqrt{1-y^2}} dy$$

$u = 1 - y^2$

$du = -2y \, dy$

$-\frac{1}{2} du = y \, dy$

$$= y \sin^{-1} y - \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{2}\right) du$$

$$= y \sin^{-1} y + \frac{1}{2} \int u^{-\frac{1}{2}} du$$

$$= y \sin^{-1} y + \cancel{\frac{1}{2}} \cdot \cancel{2} u^{\frac{1}{2}} + C$$

$$= y \sin^{-1} y + (1-y^2)^{\frac{1}{2}} + C = \boxed{y \sin^{-1} y + \sqrt{1-y^2} + C}.$$

④ $\int e^{2x}\cos(3x)\,dx$

Integration by Parts

$u = e^{2x}$ $\quad dv = \cos(3x)\,dx$

$du = 2e^{2x}dx$ $\quad V = \frac{1}{3}\sin(3x)$

$$= e^{2x}\frac{1}{3}\sin(3x) - \int \frac{1}{3}\sin(3x)\cdot 2e^{2x}\,dx$$

$$= \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\underline{\int \sin(3x)\cdot e^{2x}\,dx}$$

Integration by Parts once more

$u = e^{2x}$ $\quad dv = \sin(3x)\,dx$

$du = 2e^{2x}dx$ $\quad V = -\frac{1}{3}\cos(3x)$

$$= \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\left[e^{2x}\left(-\frac{1}{3}\cos(3x)\right) - \int -\frac{1}{3}\cos(3x)\cdot 2e^{2x}\,dx\right]$$

$$= \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\cdot\left(-\frac{1}{3}\right)e^{2x}\cos(3x) + \frac{2}{3}\int -\frac{2}{3}\cos(3x)e^{2x}\,dx$$

Thus, we have

$$\underline{\int e^{2x}\cos(3x)\,dx} = \frac{1}{3}e^{2x}\sin(3x) + \frac{2}{9}e^{2x}\cos(3x) - \underline{\frac{4}{9}\int \cos(3x)e^{2x}\,dx}$$

$$\left(1+\frac{4}{9}\right)\int e^{2x}\cos(3x)\,dx = \frac{1}{3}e^{2x}\sin(3x) + \frac{2}{9}e^{2x}\cos(3x)$$

$$\frac{13}{9}\int e^{2x}\cos(3x)\,dx = \frac{1}{3}e^{2x}\sin(3x) + \frac{2}{9}e^{2x}\cos(3x)$$

Therefore,

$$\boxed{\int e^{2x}\cos(3x)\,dx = \frac{9}{13}\left(\frac{1}{3}e^{2x}\sin(3x) + \frac{2}{9}e^{2x}\cos(3x)\right) + C.}$$