DIVISIBILITY TESTS UNIFIED: STACKING THE TRIMMINGS FOR SUMS

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SUMMARY Divisibility Tests are algorithms that can quickly decide if one integer is divisible by another. There are many tests, but at their heart most are either of the *trimming* or *summing* variety. Our aim is to derive a trimming test for every integer and to unify the trimming and summing families of tests by showing that the best summing tests can be derived directly from the trimming tests. We introduce the notion of *stacking* and use only the most basic of divisibility properties to achieve our goal.

Divisibility tests have their ancient roots in the Talmud. Through the last millennium, they have claimed the attentions and affections of such mathematical luminaries as Fibonacci, Pascal, Lagrange and Sylvester, and there are numerous such tests catalogued in Chapter XII of Dickson's monumental *History of the Theory of Numbers* [3]. The tests that many of us were fortunate enough to encounter in elementary school were the *last-digit* test for divisibility by 2, 5 or 10, the *sum of the digits* tests for 3 and 9 and the *alternating sum of the digits* test for 11. These algorithms circumvented the pedantry of long division and enchanted our fellow budding mathematicians even if the modern computer era negated the practical need for such tests.

In our more mature years as college freshmen we proved some of these tests, the last-digits without fuss and the others with our first applications of the binomial theorem and modular arithmetic. We were then presented with other tests, like that for 7, which stated that an integer a, written in the form $a = 10\bar{a} + a_0$, is divisible by 7 if and only if 7 divided the number $\bar{a} - 2a_0$. This *trimming* procedure, the given a being "trimmed" to another with one less digit, was seemingly cut from a different cloth from their better known *summing* brethren. We were poor in mathematical knowledge in those days, but we were happy with a picture of mathematics as a collection of problems to be solved. Trimming for one test, summing for another, bothered us not a jot. It is only in later years that we insist on a coherent narrative rather than a collection of vignettes and it is with this more mature sensibility that we return to the enchanting divisibility tests of our youth.

Our goal here is to provide one simple divisibility test for all integers not already covered by the well known last-digits tests. We do so by extending the trimming tests for 7, and we will show that these extended trimming tests provide summing tests for all integers. The trimming tests were explicitly described by Rina Zazkis [9] in 1999, although similar tests were proposed in 1876 by the great visual geometer Victor Schlegel [3, p. 340]. The summing tests were constructed via modular arithmetic by Apoorva Khare [4] in 1997. While Zazkis' and Khare's tests deserve to be known

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for their own sake, we will show how to derive the summing tests by *stacking* the trimming tests. To the best of our knowledge this connection between the two families of tests was not previously known.

In homage to the school venue where many of us were first exposed to these tests, we will only require basic divisibility properties with a dash of the induction axiom. The notion of *stacking* is little more than the claim that a six year old would always choose 10 pennies over a dime. We will not use anything as advanced as the binomial theorem or modular arithmetic. The well known sum and alternating sum of digits tests for 9 and 11 follow as corollaries, yet the summing tests are different and better (in a well defined sense) than the other summing tests derived from the binomial theorem.

Defining Divisibility Tests

We could operate under a Justice Potter-like assumption [7] that *we all know a divisibility test when we see it*, but let us instead formulate a definition using our favorite childhood tests as a guide. Taking a fixed $a = 2184$ as our running example, these tests for a are as follows: a is easily seen to be divisible by $q = 2$, but not 5 or 10, since a's last digit is 4. The last two digits of a are 84 and so a is easily seen to be divisible by 4. The sum of a's digits is $2 + 1 + 8 + 4 = 15$ implying that a is divisible by 3 but not 9. The alternating sum is $2 - 1 + 8 - 4 = 5$, implying that 2184 is not divisible by 11. These tests can be found in many "Introduction to Proof" texts [2, 6].

What are the characteristic features of these tests? In most basic terms a *divisibility test for an integer* q is a function from the integers to the integers, $f_q : \mathbb{Z} \to \mathbb{Z}$, such that for every integer a, q divides a if and only if q divides $f_q(a)$. For example, the tests $f_2 = f_5 = f_{10}$ mapped a to its last digit; f_4 to its last two digits; f_3 and f_9 to the sum of the digits of a; f_{11} to the alternating sum of a's digits. The identity function is such a function, but we really want a little more. Specifically, a divisibility test $f_q(a)$ should be *easy* to compute and it should be *easier* to decide if q divides $f_q(a)$ than if q divides a.

Granted, the terms "easy" and "easier" are a little hard to quantify. One reasonable criterion for "easy" is that these functions are mentally computable with relative ease. As for "easier," it could be that the number of digits in $f_q(a)$ is less than that in a for most integers a. A last observation is that any test f_q should be iterable. That is, $f_q^2(a) = f_q(f_q(a))$ is also a test for q dividing a, as is $f_q^3(a) = f_q(f_q(f_q(a)))$, etc.

The Basic Tool Chest and Last-Digit Test

For completeness, and to set the table for future tests, let us prove the divisibility tests for powers of 2 and 5. First, we promised that we would only use basic divisibility properties to derive all tests. These properties are as follows:

(1) *If two integers r and s are both divisible by q then their sum and difference, $r \pm s$ are also divisible by q.*

(2) *Any integer q with last digit q_0 equal to $1, 3, 7$ or 9 is relatively prime to 10.*

We know that these basic claims are well known to the reader and appear in every introductory number theory text like [1, §2.2]. We state these properties so we can cite them in later arguments.

Next, we can write an integer a as $a = a_n a_{n-1} \ldots a_2 a_1 a_0 = \sum_{k=0}^{n} 10^k \cdot a_k$. The *length* of a is $n + 1$ and is denoted by length(a). Letting $a_{[k,l]} := a_k a_{k-1} \ldots a_{l+1} a_l$, we can always write $a = 10^k \cdot a_{[n,k]} + a_{[k-1,0]}$. As a special case, letting $a_{[n,1]}$ be denoted by $\bar{a}$, we write

$$a = 10 \cdot \bar{a} + a_0 \quad \text{and similarly,} \quad q = 10 \cdot \bar{q} + q_0.$$

For example, if $a = 2184$ then $a_3 = 2, a_2 = 1, a_1 = 8$ and $a_0 = 4$. The length of a is 4. We can write 2184 in a variety of ways including $10^2 \cdot a_{[3,2]} + a_{[1,0]} = 10^2 \cdot 21 + 84$ and $10 \cdot \bar{a} + a_0 = 10 \cdot 218 + 4$.

Proposition 1. *The function $f_{2^k}(a) = f_{5^k}(a) = a_{[k-1,0]}$ is a divisibility test for $q = 2^k$ and $q = 5^k$.*

Proof. Since 2^k divides 10^k we see, by claim **(1)**, that 2^k divides a if and only if 2^k divides $a - 10^k \cdot a_{[n,k]} = a_{[k-1,0]}$. In other words, the last k digits of a is a test for 2^k. Since 5^k also divides 10^k, the same function is also a divisibility test for $q = 5^k$. □

For example, 8 divides $a = 2184$ because 8 divides a's last three digits , $f_{2^3}(2184) = 184$. We now turn our attention to tests for q's that are not powers of 2 or 5.

Trimming Tests for 7

We mentioned earlier that Dickson's list includes a divisibility test from the Talmud [3, p. 337]. It is a test for $q = 7$ given by $\mathrm{Tal}_7(a) = 2 \cdot a_{[n,2]} + a_1 a_0$. For example, $\mathrm{Tal}_7(2184) = 2 \cdot 21 + 84 = 126$. This can be repeated to give $\mathrm{Tal}_7(126) = 2 \cdot 1 + 26 = 28$. We can only hypothesize that Tal_7 was included in this sacred text because of the mythological and numerical prominence of 7 in the Torah. To verify this test, we mine claim **(1)** a little more: 98 is a multiple of 7, so 7 divides $a = 100 \cdot a_{[n,2]} + a_1 a_0$ if and only if it divides $a - 98 \cdot a_{[n,2]} = 2 \cdot a_{[n,2]} + a_1 a_0$. Note that 49 divides 98 too so Tal_7 is also a test for $q = 49$.

Reading through the references in Dickson, what is striking is that there can be many divisibility tests for a given fixed q. Another test for 7 was first proposed by Zbikowski [3, p. 339] in 1859: $T_7(a) = \bar{a} - 2 \cdot a_0$. For example, $T_7(2184) = 218 - 2 \cdot 4 = 210$. Note that T_7 takes a given a and "trims" it to another integer of length one less than the original a. This motivates us to define a trimming test as follows:

(Trimming) *A divisibility test f_q is called a trimming test if the length of $f_q(a)$ is one less than the length of a, for almost any a.*

We say "almost" because if a is already a single digit there is nothing to be done, and there can be instances, like $T_7(49) = -14$, where the test takes a two-digit number to another two-digit number. We wish to generalize the trimming test T_7 to provide trimming tests T_q for any q.

Let us give a concrete motivation for why T_7 works, for the special case when $a = 2184$. We know by claim **(1)** that if we add or subtract a multiple of 7 from 2184 the result is divisible by 7 if and only if 2184 is divisible by 7. We make an astute choice for the subtracted number, 21 times the last digit of 2184, namely $21 \cdot 4$, to yield $2184 - 21 \cdot 4$. Before calculating the result of this subtraction, observe that it equals $(10 \cdot 218 + 4) - ((20 + 1) \cdot 4)$. The 4's cancel, leaving the last digit of the difference as zero. In other words, what results is a multiple of 10. By claim **(2)**, 10 is relatively prime to 7, so for the purposes of testing divisibility we can trim that right-most zero from $2184 - 21 \cdot 4 = 2100$ to get 210. Thus, our decision of whether 7 divides 2184 becomes equivalent to deciding if 7 divides 210. We can extend this result for a and any q with last digit equal to 7.

Proposition 2. *For every q with $q_0 = 7$ the function $T_q(a) := \bar{a} - (3\bar{q} + 2)a_0$ is a trimming test.*

Proof. Observe that the last digit of $3q$ equals 1 and consider the difference $a - 3qa_0$:

$$\begin{aligned} a - 3qa_0 &= (10\bar{a} + a_0) - 3(10\bar{q} + 7)a_0 = 10(\bar{a} - 3\bar{q}a_0) + (1 - 3 \cdot 7)a_0 \\ &= 10(\bar{a} - 3\bar{q}a_0) - 20a_0 = 10 \cdot (\bar{a} - (3\bar{q} + 2)a_0). \end{aligned}$$

By claim **(1)**, q divides a if and only if q divides $a - 3qa_0$. By claim **(2)**, q is relatively prime to 10 and so q divides $a - 3qa_0$ if and only if it divides $\bar{a} - (3\bar{q}+2)a_0$. Hence $T_q(a) := \bar{a} - (3\bar{q}+2)a_0$ is a test. It is a trimming test since $T_q(a)$ is one digit shorter than a whenever $\text{length}(a) > \text{length}(q) + 1$. □

The first examples, after $q = 7$, of these tests are:

$q_0 = 7$: If $q = 17$ then $\bar{q} = 1$ and $T_{17}(a) = \bar{a} - (3 \cdot 1 + 2)a_0 = \bar{a} - 5a_0$.
For $a = 2184$, $T_{17}(2184) = 218 - 5 \cdot 4 = 198$ and $T_{17}(198) = -21$.
If $q = 27$ then $\bar{q} = 2$ and $T_{27}(a) = \bar{a} - (3 \cdot 2 + 2)a_0 = \bar{a} - 8a_0$.
For $a = 2184$, $T_{27}(2184) = 218 - 8 \cdot 4 = 186$ and $T_{27}(186) = -30$.
If $q = 37$ then $\bar{q} = 3$ and $T_{37}(a) = \bar{a} - (3 \cdot 3 + 2)a_0 = \bar{a} - 11a_0$.
For $a = 2184$, $T_{37}(2184) = 218 - 11 \cdot 4 = 174$ and $T_{37}(174) = -27$.

With these trimming tests in hand, we turn to constructing tests for all q not covered by the last-digits test.

Generalizing the Trimming Test for 7

The trimming tests T_q with last digit $q_0 = 7$ were premised on one central idea, that of creating a multiple of q which, when subtracted from a, results in a trimmable last digit of 0. In the case of $q_0 = 7$, the choice of $c = 3$ led to cq having 1 as its last digit. To extend this technique to $q_0 = 1, 3$ or 9, we again make an astute choice of a constant c_q for each q. To determine c_q, consider the difference $a - c_q q a_0$:

$$\begin{aligned} a - c_q q a_0 &= (10\bar{a} + a_0) - c_q(10\bar{q} + q_0)a_0 \\ &= 10\bar{a} - 10c_q\bar{q}a_0 - (c_q q_0 - 1)a_0 = 10 \cdot \left(\bar{a} - c_q\bar{q}a_0 - \frac{c_q q_0 - 1}{10} \cdot a_0\right). \end{aligned}$$

The choices of $c_q = 1, 7, 3, 9$ for $q_0 = 1, 3, 7, 9$ respectively give $c_q q_0$'s last digit equal to 1, which implies that 10 divides $c_q q_0 - 1$. These choices create the following tests originally proposed by Schlegel, which we denote by $\mathrm{Sch}_q(a)$:

Proposition 3. *There is a trimming test for every q with last digit $1, 3, 7$ or 9:*

q_0	1	3	7	9
c_q	1	7	3	9
$\mathrm{Sch}_q(a)$	$\bar{a} - \bar{q}a_0$	$\bar{a} - (7\bar{q} + 2)a_0$	$\bar{a} - (3\bar{q} + 2)a_0$	$\bar{a} - (9\bar{q} + 8)a_0$

Proof. By claim **(1)**, q divides a if and only if it divides $a - c_q q a_0$. By claim **(2)**, this is equivalent to $\frac{c_q q_0 - 1}{10}$ being an integer. We will prove the $q_0 = 9$ case. Choosing $c_q = 9$ gives $\frac{c_q q_0 - 1}{10} = \frac{9 \cdot 9 - 1}{10} = 8$. The term $c_q\bar{q}a_0 = 9\bar{q}a_0$. Substituting terms in the expression $\bar{a} - c_q\bar{q}a_0 - \frac{c_q q_0 - 1}{10} \cdot a_0$, gives the test $\mathrm{Sch}_q(a) = \bar{a} - (9\bar{q} + 8)a_0$ for all q with $q_0 = 9$.

The $q_0 = 7$ case was already shown in Proposition 2 and we leave the other two cases for the reader to verify. □

As expected, $\mathrm{Sch}_q(a) = T_q(a)$ when $q_0 = 7$. These are decent tests but we can do better. In particular, the Schlegel tests for q with last digit $q_0 = 9$ are problematic in three ways. First, $\mathrm{Sch}_9(a) = \bar{a} - 8a_0$ is not a test that captures the mathematical heart like our childhood sum of the digits test for 9. Second, looking at the next such Schlegel test, $\mathrm{Sch}_{19}(a) = \bar{a} - 17a_0$, mentally multiplying 17 and then subtracting from another number is not an *easy* task. Third, subtracting a multiple of 17 from a, as opposed to 19 itself, is not advantageous. The tests $\mathrm{Sch}_{29}(a) = \bar{a} - 26a_0$ and $\mathrm{Sch}_{39}(a) = \bar{a} - 35a_0$ are similarly problematic.

However, if we were to add qa_0 to each of these examples we would have still have divisibility tests: $\mathrm{Sch}_9(a) + 9a_0 = \bar{a} + a_0$, $\mathrm{Sch}_{19}(a) + 19a_0 = \bar{a} + 2a_0$, and $\mathrm{Sch}_{29}(a) + 29a_0 = \bar{a} + 3a_0$, etc. These do not have the same shortcomings as the original Schlegel tests yet we did something counterintuitive here: we added, rather than subtracted, a quantity to the original a. Perhaps we should pin our hopes on the adage of giving a little to get a lot back.

For the tests $\mathrm{Sch}_q(a)$ when $q_0 = 9$ the coefficient of a_0, $9\bar{q}+8$, is too big to be easily computable. Let us exercise a little more care, with the aim of keeping the coefficient of a_0 in the expression $a - c_q q a_0$ as small as possible, subject to the requirement that $c_q q_0 - 1$ be divisible by 10.

Theorem 4. (Zazkis, [9]) *There is another trimming test for every q with last digit $1, 3, 7$ or 9:*

q_0	1	3	7	9
c_q	1	-3	3	-1
$T_q(a)$	$\bar{a} - \bar{q}a_0$	$\bar{a} + (3\bar{q}+1)a_0$	$\bar{a} - (3\bar{q}+2)a_0$	$\bar{a} + (\bar{q}+1)a_0$

Proof. The proof differs from Proposition 3 in our choice of c_q only. For $cq_0 - 1$ to be divisible by 10, $c_q q_0$ is either positive with last digit equal to 1, or negative with last digit equal to 9.

For the $q_0 = 9$ case, choosing $c_q = -1$ (as opposed to $c_q = 9$ for the Schlegel tests) gives $\frac{c_q q_0 - 1}{10} = \frac{-1\cdot 9 - 1}{10} = -1$. The term $c_q \bar{q} a_0 = -\bar{q}a_0$. Substituting these two terms in the expression $\bar{a} - c_q \bar{q} a_0 - \frac{c_q q_0 - 1}{10} \cdot a_0$, gives the test

$$T_q(a) = \bar{a} - (-\bar{q} - 1)a_0 = \bar{a} + (\bar{q}+1)a_0,$$

for all q with $q_0 = 9$. We leave it to the reader to verify that the choices of $c_q = 1, -3$ and 3 for $q_0 = 1, 3$ and 7 respectively imply $\frac{c_q q_0 - 1}{10} = 0, -1$ and 2 respectively and, furthermore, these respective choices generate the tests described in the table. $\square$

These general trimming tests T_q were stated by Zazkis [9] in 1999. They agree with the Schlegel tests for $q_0 = 1$ and 7, but differ when $q_0 = 3$ or 9. Note that $T_q = \mathrm{Sch}_q(a) + qa_0$ when $q_0 = 9$. So T_q is precisely the test on which we pinned our hopes above.

Some examples, besides the $q_0 = 7$ case listed earlier, are as follows:

$q_0 = 1$: If $q = 11$ then $\bar{q} = 1$ and $T_{11}(a) = \bar{a} - 1 \cdot a_0 = \bar{a} - a_0$.
For $a = 2184$, $T_{11}(2184) = 218 - 4 = 212$ and $T_{11}(212) = 19$.
If $q = 21$ then $\bar{q} = 2$ and $T_{21}(a) = \bar{a} - 2 \cdot a_0 = \bar{a} - 2a_0$.
For $a = 2184$, $T_{21}(2184) = 218 - 2 \cdot 4 = 210$.
If $q = 31$ then $\bar{q} = 3$ and $T_{31}(a) = \bar{a} - 3 \cdot a_0 = \bar{a} - 3a_0$.
For $a = 2184$, $T_{31}(2184) = 218 - 3 \cdot 4 = 206$ and $T_{31}(206) = 2$.

$q_0 = 3$: If $q = 3$ then $\bar{q} = 0$ and $T_3(a) \ = \ \bar{a} + (3 \cdot 0 + 1)a_0 \ = \ \bar{a} + a_0$.
For $a = 2184$, $T_3(2184) = 218 + 1 \cdot 4 = 222$ and $T_3(222) = 24$.
If $q = 13$ then $\bar{q} = 1$ and $T_{13}(a) \ = \ \bar{a} + (3 \cdot 1 + 1)a_0 \ = \ \bar{a} + 4a_0$.
For $a = 2184$, $T_{13}(2184) = 218 + 4 \cdot 4 = 234$ and $T_{13}(234) = 39$.
If $q = 23$ then $\bar{q} = 2$ and $T_{23}(a) \ = \ \bar{a} + (3 \cdot 2 + 1)a_0 \ = \ \bar{a} + 7a_0$.
For $a = 2184$, $T_{23}(2184) = 218 + 7 \cdot 4 = 246$ and $T_{23}(246) = 52$.
If $q = 33$ then $\bar{q} = 3$ and $T_{33}(a) \ = \ \bar{a} + (3 \cdot 3 + 1)a_0 \ = \ \bar{a} + 10a_0$.
For $a = 2184$, $T_{33}(2184) = 218 + 10 \cdot 4 = 258$ and $T_{33}(258) = 65$.

$q_0 = 7$: See Examples after Proposition 2.

$q_0 = 9$: If $q = 9$ then $\bar{q} = 0$ and $T_9(a) \ = \ \bar{a} + (0 + 1)a_0 \ = \ \bar{a} + a_0$.
For $a = 2184$, $T_9(2184) = 218 + 1 \cdot 4 = 222$ and $T_9(222) = 24$.
If $q = 19$ then $\bar{q} = 1$ and $T_{19}(a) \ = \ \bar{a} + (1 + 1)a_0 \ = \ \bar{a} + 2a_0$.
For $a = 2184$, $T_{19}(2184) = 218 + 2 \cdot 4 = 226$ and $T_{19}(226) = 34$.
If $q = 29$ then $\bar{q} = 2$ and $T_{29}(a) \ = \ \bar{a} + (2 + 1)a_0 \ = \ \bar{a} + 3a_0$.
For $a = 2184$, $T_{29}(2184) = 218 + 3 \cdot 4 = 230$ and $T_{29}(230) = 23$.
If $q = 39$ then $\bar{q} = 3$ and $T_{39}(a) \ = \ \bar{a} + (3 + 1)a_0 \ = \ \bar{a} + 4a_0$.
For $a = 2184$, $T_{39}(2184) = 218 + 4 \cdot 4 = 234$ and $T_{39}(234) = 39$.

As we can see in these examples, the tests trim one integer per iteration. The examples above are for q's with two digits but q can be of any length. For example, $T_{181} = \bar{a} - 18 \cdot a_0$ is a test for $q = 181$ from the table of Theorem 4.

We close this section on trimming tests by reducing the four trimming cases to one single test. If $q_0 = 1$ or 9 then T_q multiplies the a_0 by a term with absolute value that is approximately equal to one-tenth of q. If $q_0 = 3$ or 7 then it is approximately one-third of q. We can also see that $T_3 = T_9$, $T_{13} = T_{39}$ and $T_7 = T_{21}$. It is a nice exercise to show that:

(3) If $q_0 = 3$ or 7 then $T_q(a) = T_{3q}(a)$.

Thus, rather than four tests we only need two tests T_q, namely those tests for $q_0 = 1$ and $q_0 = 9$. Furthermore, with $[x]$ denoting the nearest integer to x and $\text{sign}(x)$ denoting x's sign, we can reduce these two tests to one test, leaving it to the reader to confirm:

(4) $$T_q(a) \ = \ \bar{a} + \omega_q \cdot a_0 \ \text{ where } \ \omega_q \ = \left\{ \begin{array}{rl} -\left[\frac{q}{10}\right] & \text{if} \quad q_0 = 1 \\ \left[\frac{q}{10}\right] & \text{if} \quad q_0 = 9 \end{array} \right\} = \text{sign}\left(10 \cdot \left[\tfrac{q}{10}\right] - q\right) \cdot \left[\tfrac{q}{10}\right].$$

In summary, the four families of trimming tests really boil down to one trimming test. For example, for $q = 39$, $\omega_{39} = \text{sign}\left(10 \cdot \left[\frac{39}{10}\right] - 39\right) \cdot \left[\frac{39}{10}\right] = \text{sign}\,(40 - 39) \cdot 4 = +4$ and $T_{39} = \bar{a} + \omega_{39} a_0 = \bar{a} + 4a_0$. With the general trimming test, $T_q(a) \ = \ \bar{a} + \omega_q \cdot a_0$ we shall generate a generalized sum of the digits test for all q. We first motivate and add one more tool to our chest, that of *stacking*.

Stacking: Preferring Pennies to Dimes

The trimming tests $T_9(a) = \bar{a} + a_0$ and $T_{11}(a) = \bar{a} - a_0$ are not the same, but they look similar to the sum and alternating sum of digits tests respectively. The sum of the digits tests are usually verified by modular arithmetic (geometric series suffice too), but the trimming tests have only used the basic divisibility properties **(1)** and **(2)**. We will show that the trimming tests T_q not only imply the usual tests for 9 and 11, but also lead to a *summing test* for every q.

Let us focus first on the trimming test $T_9(a) = \bar{a} + a_0$ with our running example $a = 2184$. While the sum of digits test is $2184 \longrightarrow 2 + 1 + 8 + 4 = 15$, the trimming test applied iteratively is

$$2184 \xrightarrow{T_9} 218 + 4 = 222 \xrightarrow{T_9} 22 + 2 = 24 \xrightarrow{T_9} 2 + 4 = 6.$$

These tests are in the same spirit, but are superficially different. We claim that they are in fact the same by availing of a procedure that we will call "stacking."

We take for granted a non-trivial theorem [1, §1.2]: *Every positive integer has a unique base* 10 *representation.* Colloquially and in contrast, we are allowed to express 2184 in unambiguous but different ways, as both"two thousand, one hundred and eighty four" and as "twenty one hundred and eighty four." In other words, the former is in keeping with mathematical custom, but we all know that the latter means the same thing, even if the coefficient (if we are allowed to call it that) of one hundred in the latter is not between 0 and 9.

Let us think about other colloquial ways that we can bend unique representability. When adults add two integers, like the sum $218 + 4$ that resulted from $T_9(2184)$, they always write 222. We are trained to simplify with respect to unique representability base 10. Computing the sum $218 + 4$ is equivalent to giving an adult 218 cents as 21 dimes and 8 pennies and giving them a further 4 pennies. With these coins, the adult prefers to replace the $8 + 4 = 12$ pennies with 1 dime and 2 pennies, a total of 22 dimes and 2 pennies. We are trained to value efficiency in our currency and the fewer we have for the same monetary value the better.

However, when I gave my six-year old daughter the same choice, she instead opted to keep the 12 pennies. She knows that 10 pennies equals 1 dime equals a total of 10 cents, but 10 pennies are far more fun to play with than a dime, and she chose to stack the pennies together. In other words, she opts for $218 + 4 = 10 \cdot 21 + 8 + 4 = 10 \cdot 21 + (8 + 4)$. Verbally this is "21 dimes and $8 + 4 = 12$ pennies." We refer to this process of stacking 1's together as simply *stacking*, that is:

(Stacking) *Given an integer* $r = 10\bar{r} + r_0$ *and a (possibly empty) sum of integers* s *write the stacking of their sum* $r + s \overset{\text{Stack}}{=\!=} 10\bar{r} + (r_0 + s)$.

The stacking procedure is *not* a function, rather a different way of representing a sum of integers. Since it will be a very helpful way of rewriting the sum of two numbers, please permit us an abuse of notation by using "Stack" in a way that looks like it is a function. For example, stacking 218 and 4 together equals the representation $10 \cdot 21 + 12$ of 222. We will write this as $\text{Stack}(218+4) = 10 \cdot 21 + (8+4)$. Of greatest importance, since stacking is nothing more than an alternative representation of $r+s$, q divides $r+s$ if and only if q divides $\text{Stack}(r+s)$.

Stacking the Trimmings for Summing Tests

With stacking in mind, iteratively trim as before with T_9 but follow each trimming with a stacking. Note that when we defined the trimming tests $T_q(a) = \bar{a} + \omega_q a_0$ we never made any implicit assumption about a_0 being between 0 and 9 so we can always regard a stacking of 1's to be the last digit, even if there are 10 or more of such 1's stacked:

$$2184 \xrightarrow{T_9} 218 + 4 \xlongequal{\textbf{Stack}} 10 \cdot 21 + (8+4)$$
$$\xrightarrow{T_9} 21 + (8+4) \xlongequal{\textbf{Stack}} 10 \cdot 2 + (1+8+4)$$
$$\xrightarrow{T_9} 2 + (1+8+4) \xlongequal{\textbf{Stack}} (2+1+8+4)$$

Recalling the request to abuse notation for stacking, we denote the trimming of given integer a followed by the stacking of the result as $\text{Stack} \circ T_9(a)$. The above says that $(\text{Stack} \circ T_9)^3(2184) = (2+1+8+4) = 15 =: S_9(2184)$, where the latter denotes the usual sum of the digits test for 9. Let us see if iteratively trimming and stacking can also provide a sum-like test for $q = 7$. We shall use our running example $a = 2184$ again, recalling that $T_7(a) = \bar{a} + (-2)a_0$.

$$2184 \xrightarrow{T_7} 218 + (-2) \cdot 4 \xlongequal{\textbf{Stack}} 10 \cdot 21 + (8 + (-2) \cdot 4)$$
$$\xrightarrow{T_7} 21 + (-2) \cdot (8 + (-2) \cdot 4)$$
$$\xlongequal{\textbf{Stack}} 10 \cdot 2 + (1 + (-2) \cdot (8 + (-2) \cdot 4))$$
$$\xrightarrow{T_7} 2 + (-2) \cdot (1 + (-2) \cdot (8 + (-2) \cdot 4))$$
$$\xlongequal{\textbf{Stack}} 2 + (-2) \cdot (1 + (-2) \cdot (8 + (-2) \cdot 4)).$$

In other words $(\text{Stack} \circ T_7)^3(2184) = 2 + (-2)^1 \cdot 1 + (-2)^2 \cdot 8 + (-2)^3 \cdot 4$, suggesting (not proving, mind) an unusual sum of digits type test, $\sum_{j=0}^{n} (-2)^j a_{n-j}$, for $q = 7$.

(Summing) *A divisibility test* $f_q(a) = \sum_{j=0}^{n} \gamma_j a_{n-j}$ *is called a summing test for* q *if each* $\gamma_j \in \mathbb{Z}$.

The above examples for $q = 7$ and $q = 9$ with $a = 2184$ suggest $\gamma_j = (-2)^j = \omega_7^j$ and $\gamma_j = 1 = \omega_9^j$ generate summing tests for 7 and 9 respectively. We claim this holds in general.

Theorem 5. (essentially due to Khare [4]) *If* $T_q = \bar{a} + \omega_q a_0$ *is a trimming test for* q *then* $S_q(a) := \sum_{j=0}^{n} \omega_q^j a_{n-j}$ *is a summing test for* q.

These tests were first presented by Khare [4] in 1997. Her modular arithmetic proof does not involve trimming tests and her ω_q's are otherwise defined. We will present Khare's proof later. Our proof uses neither modular arithmetic or the binomial theorem and it unifies the trimming and summing families. First, some examples for $a = 2184$:

$$\begin{array}{llll}
q = 7: & \omega_7 = -2 & \text{and} & S_7(2184) \;=\; 2 + (-2)\cdot 1 + (-2)^2 \cdot 8 + (-2)^3 \cdot 4 = 0. \\
q = 9: & \omega_9 = 1 & \text{and} & S_9(2184) \;=\; 2 + (1)\cdot 1 + (1)^2 \cdot 8 + (1)^3 \cdot 4 = 0. \\
q = 11: & \omega_{11} = -1 & \text{and} & S_{11}(2184) \;=\; 2 + (-1)\cdot 1 + (-1)^2 \cdot 8 + (-1)^3 \cdot 4 = 5. \\
q = 17: & \omega_7 = -5 & \text{and} & S_{17}(2184) \;=\; 2 + (-5)\cdot 1 + (-5)^2 \cdot 8 + (-5)^3 \cdot 4 = -303. \\
q = 39: & \omega_{39} = 4 & \text{and} & S_{39}(2184) \;=\; 2 + (4)\cdot 1 + (4)^2 \cdot 8 + (4)^3 \cdot 4 = 390. \\
q = 43: & \omega_{43} = 13 & \text{and} & S_{43}(2184) \;=\; 2 + (-13)\cdot 1 + (-13)^2 \cdot 8 + (-13)^3 \cdot 4 = -7447. \\
q = 91: & \omega_{91} = -9 & \text{and} & S_{91}(2184) \;=\; 2 + (-9)\cdot 1 + (-9)^2 \cdot 8 + (-9)^3 \cdot 4 = -2275.
\end{array}$$

Proof of Theorem 5 by Trimming and Stacking. We will show, by induction on the length of a, that $S_q(a) = (\text{Stack} \circ T_q)^n(a)$ whenever a has length $n+1$. If $n = 1$ then $a = a_1a_0$ has length two. In this case, $\text{Stack}(T_q(a_1a_0)) = \text{Stack}(\bar{a} + \omega_q a_0) = a_1 + \omega_q a_0$, which equals $S_q(a_1a_0)$, as claimed.

Assuming that $S_q(a') = (\text{Stack} \circ T_q)^{n-1}(a')$ whenever a' has length n, we proceed by looking at integers $a = a_na_{n-1}\ldots a_2a_1a_0$ with length $n+1$. Applying $\text{Stack} \circ T_q$ to this a results in $\text{Stack}(T_q(a)) = \text{Stack}(\bar{a} + a_0) \;=\; 10a_{[n,2]} + (a_1 + \omega_q a_0) =: a'$, an integer with n digits, with last digit equal to $(a_1 + \omega_q a_0)$. Hence,

$$\begin{aligned}
(\text{Stack} \circ T_q)^n(a) &= (\text{Stack} \circ T_q)^{n-1}(\text{Stack} \circ T_q(a)) \\
&= (\text{Stack} \circ T_q)^{n-1}(10a_{[n,2]} + (a_1 + \omega_q a_0)) \\
&= S_q(10a_{[n,2]} + (a_1 + \omega_q a_0)) \\
&= \sum_{j=0}^{n-2} \omega_q^j a_{n-j} + \omega_q^{n-1}(a_1 + \omega_q a_0) \;=\; \sum_{j=0}^{n-2} \omega_q^j a_{n-j} + \omega_q^{n-1} a_1 + \omega_q^n a_0 \\
&= \sum_{j=0}^{n} \omega_q^j a_{n-j} = S_q(a).
\end{aligned}$$

$\square$

These are wonderful tests because they generalize the sum and alternating sum of digits tests of our childhood. However, there is a practical caveat. The outputs for $S_q(a)$ become increasingly harder to compute as q gets larger and it can be harder to decide if q divides $S_q(a)$ than if q divides a. The trimming tests alone are often more manageable. For example, $S_{43}(2184) = 2 - 13 + 1352 - 8788 = -7447$ is a challenge to do in one's head and deciding if 43 divides 7447 is harder than the original $a = 2184$. In contrast, $T_{43}(2184) = 218 - 13 \cdot 4 = 218 - 52 = 166$ and $T_{43}(166) = 16 - 13 \cdot 6 = -52$ is both easier to compute and easier to test for divisibility by 43.

Contrasting with the Binomial Tests

The tests S_9 and S_{11} are the sum and alternating sum of digits tests that are so well known, with the usual method for proving the tests being the binomial theorem, or equivalently and more concisely from modular arithmetic. We assume at this juncture some knowledge of modular arithmetic along the lines of [1, Chapter 4]. We claim that Khare's tests S_q are *better* than the binomial tests B_q in the sense that for most integers a, $S_q(a)$ is much smaller and easier to compute than $B_q(a)$. We will make this more precise.

The *binomial tests* for each q are derived from writing 10^j as $(q+(10-q))^j$ and rewriting the standard expression for a modulo q as

$$a \;=\; \sum_{j=0}^{n} 10^j a_j \;\equiv\; \sum_{j=0}^{n} (10-q)^j a_j =: B_q(a) \quad \bmod q.$$

For example, $B_{39}(a) = \sum_{j=0}^{n}(-29)^j a_j$ has no benefit over the original $a = \sum_{j=0}^{n} 10^j a_j$ for testing divisibility by 39. In contrast, $S_{39}(a) = \sum_{j=0}^{n} 4^j a_{n-j}$ is a much easier test to compute since the coefficient terms $(-29)^j$ are more difficult to calculate than 4^j.

In general, $B_q(a)$'s coefficients are $(10-q)^j$ making any test difficult to compute when $q > 20$. On the other hand, $S_q(a)$'s coefficients are a fraction of q: From **(3)** and **(4)**, the coefficients ω_q for S_q are approximately of size

$$|\omega_q| \;\approx\; \begin{cases} \frac{1}{10}q & \text{if} \quad q_0 = 1 \text{ or } 9 \\ \frac{3}{10}q & \text{if} \quad q_0 = 3 \text{ or } 7 \end{cases}$$

For example, $|\omega_{39}| = 4 \approx \frac{39}{10}$. Generally, S_q is worth using whenever $q < 100$ and $q_0 = 1$ or 9, or when $q < 40$ and $q_0 = 3$ or 7. The test S_q may not be as effective for $q > 100$ but the trimming test remains appealing. For example, $T_{181}(a) = \bar{a} - 18a_0$ multiplies the last digit by 18 and trims accordingly and, in our view, is better than dividing by the three-digit number 181, even if many iterations of the trimming test are needed.

Closing Comments

Starting with the test for 7 and using only elementary tools, we generated a single trimming test for all integers and we derived a summing test for all integers from these trimming tests, adding only a dash of the induction axiom to our basic arsenal. In Tolstoyan terms [8], divisibility tests appear to be families with distinct ways but are really all alike.

We close with some short remarks, left to the reader to verify and explore. We have claimed that the trimming tests are for all integers q, with the exception of those already covered by the last-digits tests. Our trimming tests were for all integers ending in a $1, 3, 7$ or 9 and these integers were not assumed to be prime. We claim that every integer q' can be factored into a product of a power of 2, a power of 5 and a q with last digit $1, 3, 7$ or 9. As a consequence, q' divides an integer a if and only if a passes the last-digits tests for the powers of 2 and 5 and passes the trimming or summing tests for q described here.

As mentioned earlier, the summing tests S_q are due to Khare. Her proof started with the derivation of ω_q as the least residue representative [1, Chapter 9] of the inverse of 10 modulo q, the $\omega_q \equiv 10^{-1} \mod q$ of smallest size. We claim that Khare's ω_q equals the ω_q derived from the trimming test T_q of Zazkis. With this choice in hand, Khare's construction of S_q follows immediately:

$$\omega_q^n a \;=\; \sum_{j=0}^{n} \omega_q^n 10^j a_j \equiv S_q(a) \mod q.$$

This minimum choice of ω_q is reflected in the approximations of ω_q above.

Khare also generalized the base $b = 10$ to tests for q in any base b: If q and b are co-prime then the ω_q term is precisely the least residue of $\omega_q \equiv b^{-1} \mod q$, and there are last-digits tests for all factors of b. The trimming tests can also be generalized in the same manner with the same ω_q's.

In general, the trimming tests do not preserve remainders. To derive T_q we divided the difference $a - c_q q a_0$ by 10. Therefore, $a \equiv 10^{-1} T_q(a) \mod q$. Since $10^{-1} \equiv \omega_q \mod q$ then $a \equiv \omega_q T_q(a) \mod q$. Since stacking changes the representation of the number a but not a itself, then $a \equiv \omega_q^n S_q(a) \mod q$ whenever a has length $n + 1$. In contrast, the binomial tests B_q do preserve remainders, explaining another aspect of their popularity.

Finally, while most tests are of the trimming and summing variety, there are many tests that are not equivalent to those outlined here, like the Talmudic test Tal_7. A recent and lively account of divisibility tests (heavily dependent on modular arithmetic) for every $q \leq 100$ by Renault [5] is well worth a read. Of course, Chapter XII of Dickson [3] has many gems not discussed here and independently deriving each of them would make for a great senior project.

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