what is going on. XIII.8 is about the regular pentagon with side x and diagonal b (in our present notation). It is proved that

$$b : x = x : (b - x),$$

which is, after VI.16, equivalent to

$$b(b - x) = x^2.$$

Knowing this, we understand the essential meaning of II.11: to determine the side x of a regular pentagon when the diagonal b is given. But about this Euclid keeps quiet in Book II, and even in Book IV, where he constructs the pentagon.

THE CHAIR

This chair was once a student of Euclid.

The book of his laws lay on its seat.
The schoolhouse windows were open,
So the wind turned the pages
Whispering the glorious proofs.

The sun set over the golden roofs.
Everywhere the shadows lengthened,
But Euclid kept quiet about that.

(Charles Simic: Hotel Insomnia, New York: Harcourt, Brace 1992. With kind permission of the publisher.)

Remark

One thing should be said very clearly: Mathematical truth is not dependent on motivation. Euclid has taught this to many generations of mathematicians. Teaching a class is another thing. Students have every right to understand how particular steps are directed to a specific goal.

Propositions II.12, 13 are very important as seen from today, because they translate into the law of cosines. In the *Elements*, however, they are just stated and never again mentioned or applied. Let us first look for a motivation of the statements (which, clearly, Euclid does not supply). By the congruence theorem *SAS*, a triangle—and

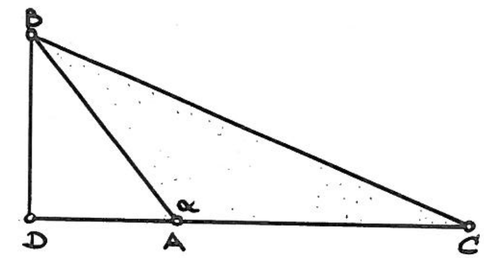

FIGURE 7.7

hence its third side—is determined by the two sides, say a, b, and the enclosed angle γ. By the theorem of Pythagoras we know the third side c if γ is a right angle. Is it possible to determine c for a general angle γ? This question is answered in II.12 for an obtuse and in II.13 for an acute angle γ. Let the first case stand for both of them.

Prop. II.12.
Let $\triangle ABC$ be an obtuse-angled triangle having $\angle BAC$ obtuse, and let BD be drawn from the point B perpendicular to CA produced [Fig. 7.7]. I say that the square on BC is greater than the squares on BA, AC together by twice the rectangle contained by CA, AD.

Proof.
By the theorem of Pythagoras I.47:

$$\square BC = \square BD + \square DC,$$
$$\square AB = \square BD + \square DA.$$

By the binomial theorem II.4,

$$\square DC = \square DA + \square AC + 2\sqsubset\!\sqsupset(DA, AC); \qquad (7.5)$$

hence

$$\square BC = \square BD + \square DA + \square AC + 2\sqsubset\!\sqsupset(DA, AC) \qquad (7.6)$$
$$= \square AB + \square AC + 2\sqsubset\!\sqsupset(DA, AC) \qquad (7.7)$$

as asserted.

Modern translation. Let $AB = c$, $BC = a$, $CA = b$, and $\angle BAC = \alpha$. Then $DA = c\cos\alpha$. Because α is obtuse, $\cos\alpha$ is negative, and we get

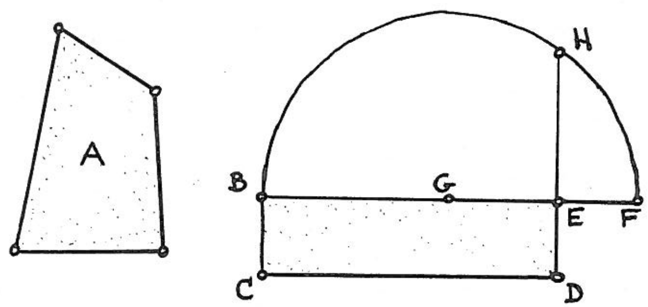

FIGURE 7.8

the result

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$

the modern version. The sign of the cosine function will take care of both cases II.12,13. The law of cosines could develop its full power only a long time after Euclid.

Prop. II.14.
To construct a square equal to a given rectilinear figure.

Solution:
Let A be the given figure [Fig. 7.8]. Construct a rectangle $\boxed{}(BE, ED)$ of equal content with A via I.45. If this rectangle is a square, we are ready. Otherwise let BE be greater than ED. Prolong BE to F and make EF equal to ED. Bisect BF in G, describe the semicircle BHF with center G. Produce DE to H.

Assertion:
$\boxed{}(BE, ED) = \square EH$.

By II.5 for the line BF we get

$$\boxed{}(BE, EF) + \square EG = \square GF.$$

GF is equal to GH by construction, hence

$$\begin{aligned}\boxed{}(BE, EF) + \square EG &= \square GH\\ &= \square EG + \square EH \qquad \text{by I.47.}\end{aligned}$$

Subtracting the square $\square EG$ on both sides and observing that $EF = EG$ completes the proof.

This theorem is a goal for its own sake. It is the objective of a theory developed from I.35–45 via I.47 and II.5 to its culminating point II.14. It serves no other purpose than to transmit a certain knowledge or answer a question that is important in theory and, we may add, in practice as well.