through the development of some very sophisticated geometric propositions beginning with just a few basic axioms and postulates.

This enchantment with building the complex from the simple was also evident in the Greeks' geometric constructions. For them, the rules of the game required that all constructions be done only with compass and (unmarked) straightedge. These two fairly unsophisticated tools—allowing the geometer to produce the most perfect, uniform one-dimensional figure (the straight line) and the most perfect, uniform two-dimensional figure (the circle)—must have appealed to the Greek sensibilities for order, simplicity, and beauty. Moreover, these constructions were within reach of the technology of the day in a way that, for instance, constructing a parabola was not. Perhaps it is accurate to suggest that the aesthetic appeal of the straight line and circle reinforced the central position of straightedge and compass as geometric tools while, conversely and simultaneously, the physical availability of these tools enhanced the role to be played by straight lines and circles in the geometry of the Greeks.

The ancient mathematicians were consequently committed to, and limited by, the output of these tools. As we shall see, even the seemingly unsophisticated compass and straightedge can produce, in the hands of ingenious geometers, a rich and varied set of constructions, from the bisection of lines and angles, to the drawing of parallels and perpendiculars, to the creation of regular polygons of great beauty. But a considerably more challenging problem in the fifth century B.C. was that of the quadrature or squaring of a plane figure. To be precise:

- The *quadrature* (or squaring) of a plane figure is the construction—using only compass and straightedge—of a square having area equal to that of the original plane figure. If the quadrature of a plane figure can be accomplished, we say that the figure is *quadrable* (or squarable).

That the quadrature problem appealed to the Greeks should come as no surprise. From a purely practical viewpoint, the determination of the area of an irregularly shaped figure is, of course, no easy matter. If such a figure could be replaced by an equivalent square, then determining the original area would have been reduced to the trivial matter of finding the area of that square.

Undoubtedly the Greeks' fascination with quadrature went far beyond the practical. For, if successfully accomplished, quadrature would impose the symmetric regularity of the square onto the asymmetric irregularity of an arbitrary plane figure. To those who sought a natural world governed by reason and order, there was much appeal in the process of replacing the asymmetric by the symmetric, the imperfect by the perfect, the irrational by the rational. In this sense, quadrature represented not only the triumph of human reason, but also the inherent simplicity and beauty of the universe itself.

Devising quadratures was thus a particularly fascinating problem for Greek mathematicians, and they produced clever geometric constructions to that end. As is often the case in mathematics, solutions can be approached in stages, by first squaring a reasonably "tame" figure and moving from there to the quadrature of more irregular, bizarre ones. The key initial step in this process is the quadrature of the rectangle, the procedure for which appears as Proposition 14 of Book II of Euclid's *Elements*, although it was surely known well before Euclid. We begin with this.

STEP 1 Quadrature of the rectangle (Figure 1.7)

Let *BCDE* be an arbitrary rectangle. We must construct, with compass and straightedge only, a square having area equal to that of *BCDE*. With the straightedge, extend line *BE* to the right, and use the compass to mark off segment *EF* with length equal to that of *ED*—that is, $\overline{EF} = \overline{ED}$. Next, bisect *BF* at *G* (an easy compass and straightedge construction), and with center *G* and radius $\overline{BG} = \overline{FG}$, describe a semicircle as shown. Finally, at *E*, construct line *EH* perpendicular to *BF*, where *H* is the point of intersection of the perpendicular and the semicircle, and from there construct square *EKLH*.

We now claim that the shaded square having side of length $\overline{EH}$—a figure we have just *constructed*—has area equal to that of the original rectangle *BCDE*.

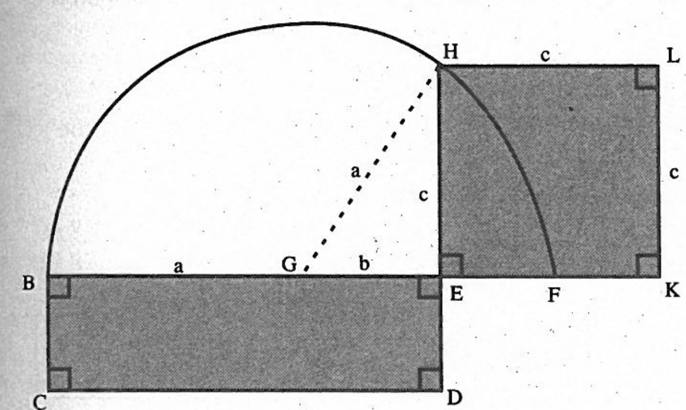

FIGURE 1.7

To verify this claim requires a bit of effort. For notational convenience, let a, b, and c be the lengths of segments HG, EG, and EH, respectively. Since $\triangle GEH$ is a right triangle by construction, the Pythagorean theorem gives us $a^2 = b^2 + c^2$, or equivalently $a^2 - b^2 = c^2$. Now clearly $\overline{FG} = \overline{BG} = \overline{HG} = a$, since all are radii of the semicircle. Thus, $\overline{EF} = \overline{FG} - \overline{EG} = a - b$ and $\overline{BE} = \overline{BG} + \overline{GE} = a + b$. It follows that

$$\begin{aligned} \text{Area (rectangle } BCDE) &= (\text{base}) \times (\text{height}) \\ &= (\overline{BE}) \times (\overline{ED}) \\ &= (\overline{BE}) \times (\overline{EF}), \text{ since we constructed } \overline{EF} = \overline{ED} \\ &= (a + b)(a - b) \text{ by the observations above} \\ &= a^2 - b^2 \\ &= c^2 = \text{Area (square } EKLH) \end{aligned}$$

Consequently, we have proved that the original rectangular area equals that of the shaded square which we *constructed* with compass and straightedge, and this completes the rectangle's quadrature.

With this done, the steps toward squaring more irregular regions come quickly.

STEP 2 Quadrature of the triangle (Figure 1.8)

Given $\triangle BCD$, construct a perpendicular from D meeting BC at point E. Of course, we call $\overline{DE}$ the triangle's "altitude" or "height" and know that the area of the triangle is ½(base) × (height) = $\frac{1}{2}(\overline{BC}) \times (\overline{DE})$. If we bisect DE at F and construct a rectangle with $\overline{GH} = \overline{BC}$ and $\overline{HJ} = \overline{EF}$, we know that the rectangle's area is $(\overline{HJ}) \times (\overline{GH}) = (\overline{EF}) \times (\overline{BC}) = \frac{1}{2}(\overline{DE}) \times (\overline{BC}) = \text{area } (\triangle BCD)$. But we then apply Step 1 to construct a square equal in area to this rectangle, and so the square's area is also that of $\triangle BCD$. This completes the quadrature of the triangle.

We next move to the following very general situation.

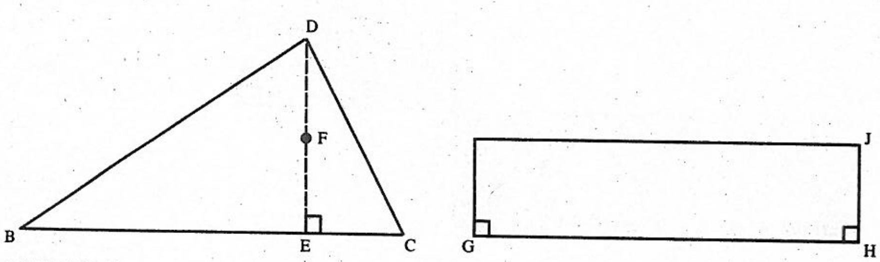

FIGURE 1.8

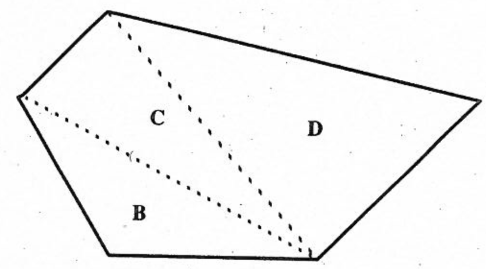

FIGURE 1.9

STEP 3 Quadrature of the polygon (Figure 1.9)

This time we begin with a general polygon, such as the one shown. By drawing diagonals, we subdivide it into a collection of triangles with areas **B**, **C**, and **D**, so that the total polygonal area is **B** + **C** + **D**.

Now triangles are known to be quadrable by Step 2, so we can construct squares with sides b, c, and d and areas **B**, **C**, and **D** (Figure 1.10). We then construct a right triangle with legs of length b and c, whose hypotenuse is of length x, where $x^2 = b^2 + c^2$. Next, we construct a right triangle with legs of length x and d and hypotenuse y, where we have $y^2 = x^2 + d^2$, and finally, the shaded square of side y (Figure 1.11).

Combining our facts, we see that

$$y^2 = x^2 + d^2 = (b^2 + c^2) + d^2 = \mathbf{B} + \mathbf{C} + \mathbf{D}$$

so that the area of the original polygon equals the area of the square having side y.

This procedure clearly could be adapted to the situation in which the polygon was divided by its diagonals into four, five, or any number of triangles. No matter what polygon we are given (see Figure 1.12), we can subdivide it into a set of triangles, square each one by Step 2, and use these individual squares and the Pythagorean theorem to build a

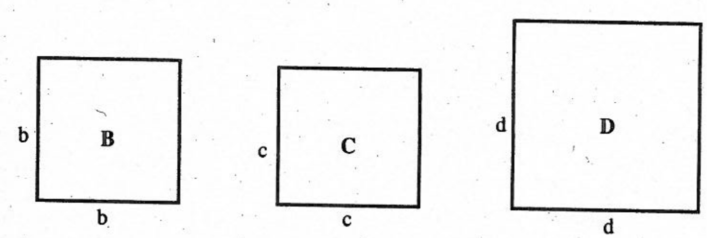

FIGURE 1.10

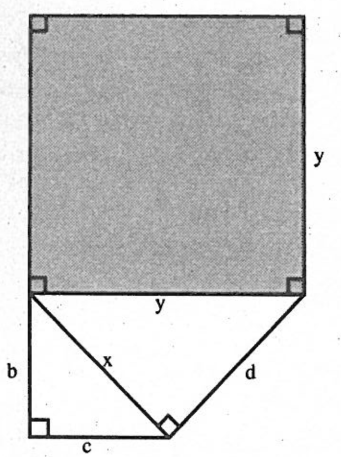

FIGURE 1.11

large square with area equal to that of the polygon. In short, polygons are quadrable.

By an analogous technique we could likewise square a figure whose area was the *difference* between—and not the sum of—two quadrable areas. That is, suppose we knew that area **E** was the difference between areas **F** and **G**, and we had already constructed squares of sides f and g with areas as shown in Figure 1.13. Then we would construct a right triangle with hypotenuse f and leg g. We let e be the length of the other leg and construct a square with side e. We then have

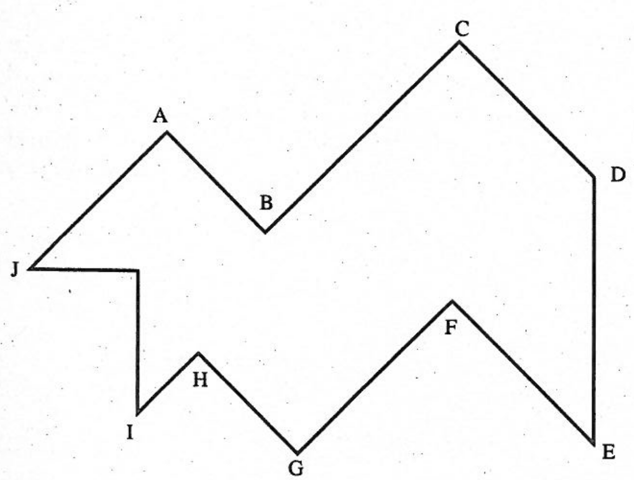

FIGURE 1.12

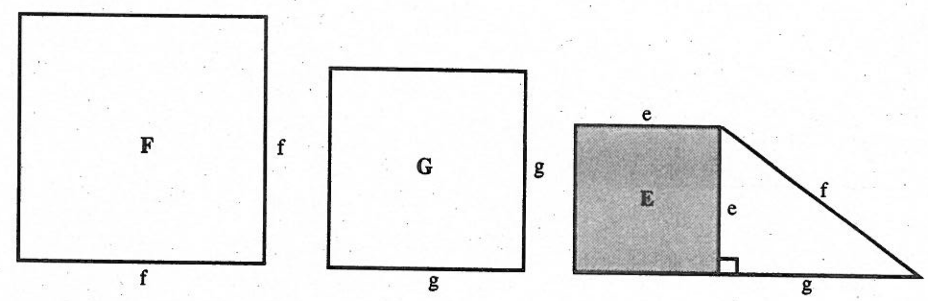

FIGURE 1.13

$$\text{Area (square)} = e^2 = f^2 - g^2 = \mathbf{F} - \mathbf{G} = \mathbf{E}$$

so that area **E** is likewise quadrable.

With the foregoing techniques, the Greeks of Hippocrates' day could square wildly irregular polygons. But this triumph was tempered by the fact that such figures are *rectilinear*—that is, their sides, although numerous and meeting at all sorts of strange angles, are merely straight lines. Far more challenging was the issue of whether figures with curved boundaries—the so-called *curvilinear* figures—were likewise quadrable. Initially, this must have seemed unlikely, for there is no obvious means to straighten out curved lines with compass and straightedge. It must therefore have been quite unexpected when Hippocrates of Chios succeeded in squaring a curvilinear figure known as a "lune" in the fifth century B.C.

Great Theorem: The Quadrature of the Lune

A lune is a plane figure bounded by two circular arcs—that is, a crescent. Hippocrates did not square all such figures but rather a particular lune he had carefully constructed. (As will be shown in the Epilogue, this distinction seemed to be the source of some misunderstanding in later Greek geometry.) His argument rested upon three preliminary results:

- The Pythagorean theorem
- An angle inscribed in a semicircle is right.
- The areas of two circles or semicircles are to each other as the squares on their diameters.

$$\frac{\text{Area (semicircle 1)}}{\text{Area (semicircle 2)}} = \frac{d^2}{D^2}$$