For the oral final exam, this list should not be viewed as exhaustive. Every question from what we've read and studied is "fair game." Nevertheless, what follows is a very decent guideline to help focus the mind in the two weeks ahead. You should consider the exercises in each section as being included.

Usually, you will not be asked to prove things in complete form but rather be asked provide a sketch of the proof and/or an illustration of how and why a given theorem works. Contrarily to what one thinks, providing a decent sketch demands a deep level of understanding of the result and the need to constantly evaluate what's straightforward and beleivable to show versus what is substantive and central to the topic at hand.

For example, if one was to be asked about why three points not on a line in $\mathbb{R}^2$ lie on a unique circle, one should give a sketch of reason by using two "bisecting" lines and, as importantly from this sketch, one should be able to say why three points that are colinear cannot lie on a circle.

The questions will closely follow the substance and style of the class tests which were very comprehensive. Below is a guide to what to expect from the final exam.

In the exam, I will ask you questions and you will use the board and/or paper; if it is clear that you know what is going on then I will move swiftly on to the next question. Don't be too perturbed by this – it is a good sign. If you are stuck I will provide some hints and you will lose some credit if you are still stuck I will provide even more hints and you will lose even more credit but not so much as to completely answer the question at hand. Please be prepared for a very rigorous exam but at the same time know that I want you to succeed and will make reasonable constructive steps to help you do so.

Axioms and Foundations in the Euclidean context

(1) How did Euclid define objects like point, line, straight line, angle, right angle, etc? And the common notions and the postulates?

(2) What are some things that Euclid should have taken as axioms but did not. On the other hand, what are examples of something that Euclid could have taken for granted but did not.

(3) The idea of motion comes up first in SAS but what are some of the propositions that Euclid proves later that can be understood as saying that Euclid didn't really want the ability to move as a postulate?

(4) What are some of the explanations for Euclid postponing use of parallels and the fifth postulate for so long? What are some of the statements that are equivalent to the fifth postulate? How do Propositions 25-29 support the use of angle in formulating Postulate 5? Can you tell us about one attempt to prove the fifth postulate?

(5) How is area introduced /defined, relative to congruencies. What roles do parallels play in this definition?

(6) What are some of the characterisitics of circles that seem obvious but that Euclid proves anyway. For example, the line joining two points on a circle is inside the circle.

(7) And speaking of congruencies, what about AAA? What can we say about AAA? Establish Thales Theorem.

(8) How is it that our notion of slope of a line (in a coordinatized plane) uses Thales Theorem? What do we define as the distance between two points. To sensibly define distance what theorem do we assume is true?

(9) In what sense does thinking about isometries and their groups permit motion?

Standard Constructions and Propositions in the Euclidean context

(1) Establish the constructions of a regular (equilateral and equiangular) triangle, square, hexagon and pentagon. What are the congruencies? Why is SSA not one?

(2) Establish various results about triangles, like [I.16]-[I.21] and the relationships between angles and lengths of sides.

(3) Establish constructions for bisecting and trisecting a line. How are bisecting lines and angles yoked? Why does this yoking not extend to trisecting angles? In what sense is trisecting an angle impossible? What extra axioms will suffice to make angle trisection possible?

(4) Establish the constructions of other lines perpendicular and parallel to a given lines through given specified points.

(5) Be able to discuss the path to the Pythagorean Theorem and its converse. In addition to the Book I treatment, provide the proof via Thales theorem too.

(6) What does it mean for a rectangle to be squareable? Indeed, establish why it is so and give a sense of what the circle is not squareble. Likewise for "doubling the cube."

(7) Various statement about properties of circles: how to find its center, and the uniqueness of the center, there being a unique circle containing three non-colinear points, how can circles cut and touch, how do the centers and touching point align, properties of tangent lines to circles. Properties of angles upon chords in circles.

(8) Show that the set of points all equidistant from two points forms a line?

(9) Establish that π is a constant. Looking at this video

`https://www.youtube.com/watch?v=72N7yjcVFC8`

(the proof in Euclid focuses in ratio of area to diameter) might give an alternative view to that of Euclid (this focuses on the ratio of circumference to diameter).

Isometries and Transformation Groups

(1) A clear sense of what is and what is not preserved by reflections.

(2) As you'd expect, establish the three reflections theorem, stating clearly the lemmas necessary to prove this result.

(3) How are reflections, rotations, translations and glide reflections established. (I must confess I am still stuck on how every isometry that requires three reflections is a glide).

(4) How does orientation fit with isometries? What kind of groups arise from those isometries that preserve orientation. How does this perspective help you classify isometries?

(5) Are reflections commutative? If not, are they ever commutative?

(6) What are the isometries on the real line? In 3-space. On the sphere? Generalize all of the above questions for the plane to these contexts.

(7) How do transformations of the Euclidean plane and projective line as 2×2 matrices. Understand Figures 7.2 and 7.3.

(8) How is it that the isometries of the sphere are effectively those of the plane. Why is it that a line on the sphere cannot be extended indefinitely, i.e., why is it that Postulate II cannot be said to hold on the sphere? What about the other Euclidean postulates?

(9) How is it that the rotations on the sphere form a group? In particular, given two specific rotations say exactly how their product can be written as a rotation in its own right.

Topics in the Development of Projective Geometry

(1) Draw the *construzione legittima* and say why one needs the compass to draw it. Say how one can construct the similar tiling "in perspective" but using straight edge alone. (§5.1 and 5.2)

(2) Discuss schematically the idea of seeing from an "all seeing eye" to model a projective line on a plane and to model a projective line by a circle. Say where the point at infinity is in each of these models. Be able to talk about the point at infinity in a meaningful way. Why is the "line" model preferred to the "circle" model?

(3) Why is it that we canot have a meaningful notion of length on projective lines (hint: what would the "distance" from the point at infinity to any other line have to be?)

(4) Justify why the cross ratio is the defining invariant of the projective line, that is, that it characterizes the LFF's and that any invariant of four points is a function of the cross ratio

(5) Define the *cross ratio* and justify why it is preserved by LFF's. Be prepared to illustrate this last claim on a specific example.

(6) Understand the three projections of projective lines of §5.5 (those of figures 5.13, 5.14, 5.15). How is it that linear fractional functions are compositions of these three projections Given an example of an LFF write it as a composition of these three projections.

Non-Euclidean Geometry

(1) Be able to say how H, the upper half of the complex plane, can be regarded as an extension of $\mathbb{R}P^1$. Say what the definition of a "line" is in H and how Postulate I holds, i.e., their exists a unique line segment between any two given points. Say also how the 5th postulate fails in H.

(2) Describe the fractional/generating transformations on H including why it is that we need to introduce the conjugate operation and how inversion in the semicircle centered at 0 and of "radius" 1 equates to the map $f(z) = \frac{1}{\bar{z}}$. In particular, be able to say why $f(z) = \frac{1}{z}$ is not sufficient for our needs.

(3) Equations of "lines" and how one can read the geometry of the line from an equation of the form $Az\bar{z} + B(z + \bar{z}) + C = 0$ (§8.2). Understand and be able to show how Möbius transformations are products of generating transformations and vice-versa and how Möbius transformations map "lines" to "lines." In particular, given a fixed line L and a transformation f explicitily describe the line $f(L)$. (§8.3 and 8.4)

(4) Say how the generating functions of the projective line extend to generating functions of H. Say how we now have a notion of reflections in H.

(5) Understand what it means to say that Möbius transformations preserve angle and be prepared to describe an angle between two lines. Say how this provides a meaningful notion of angle and thus Postulate IV can be taken as something that holds.(§8.5)

(6) Know how to motivate and define non-Euclidean distance. In particular, given an explicit pair of points compute the Euclidean distance between these two points. Why is it that every line is infinitely long and why is that the distance between, say $1/2$ and 1 is infinite? Say why Euclidean postulates II and III hold for H with this notion of length.(§8.6)

(7) Finally, understand how the three reflections theorem of Euclidean plane has its analogue in H (this is the topic of §8.7 and 8.8).

Talk to me like a sixteen year old With all the above questions, we should have careful but accessible descriptions for conversations with 16 year olds. For example, you should be able to respond in a sensible way to what we mean by an axiom, a definition, a proposition and how we distinguish between them? Can you summarize the themes of Book I in ten sentences.