CONSTRUCTING CAYLEY GRAPHS VIA TESSELATIONS
OF RIEMANN SURFACES

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Abstract. We exhibit an explicit action of the modular group on certain arithmetic Riemann Surfaces. We then show how this allows a nice characterization of certain graphs associated to these surfaces.

1. Introduction

We denote the complex upper half plane by $\mathcal{H}$. Set
\[ \Gamma = PSL(2, \mathbb{Z}) = \frac{SL(2, \mathbb{Z})}{\langle \pm 1 \rangle}. \]
Then $\Gamma$ acts on $\mathcal{H}$ via fractional linear transformations. That is, if \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), then we define
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}. \]
Two points $z_1, z_2 \in \mathcal{H}$ are said to be equivalent if there is a fractional linear transformation mapping one to the other. We note, in passing, that given $M \in SL(2, \mathbb{Z})$, the matrices $\pm M$ induce the same fractional linear transformation. It was for this reason that we passed to the quotient $PSL(2, \mathbb{Z})$.

It is well known that the region in $\mathcal{H}$ above the unit circle and between the vertical lines $\text{Re}(z) = \pm \frac{1}{2}$ is a fundamental domain for this action (see [6]). By this we mean that every point in $\mathcal{H}$ is equivalent to a point in this region, and any two distinct, equivalent points within the region must both lie on the boundary. We will refer to this region, or any region equivalent to it under the action of $\Gamma$, as a modular triangle and denote such a region by $\mathcal{T}$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and let $n \geq 2$ be a positive integer. We define the principle congruence subgroup of level $n$ of $\Gamma$, denoted by $\Gamma_n$, to be the set
\[ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}. \]
It is well known that a fundamental domain for the action of $\Gamma_n$ on $\mathcal{H}$ can be obtained by gluing together copies of $T$ along boundary edges. Furthermore, this can be done in such a way that the resulting region in $\mathcal{H}$ is simply connected. It is easily shown that $\Gamma/\Gamma_n \cong PSL(2, \mathbb{Z}_n)$, where $\mathbb{Z}_n$ denotes the ring of integers modulo $n$. Set $\mu = |PSL(2, \mathbb{Z}_n)|$. Then it is also known that $\mu$ triangles are needed to obtain a fundamental domain for the action of $\Gamma_n$. Proofs of these assertions can be found in [5].

Let $\mathcal{F}_n$ denote a fundamental region in $\mathcal{H}$ for the action of $\Gamma_n$ obtained by gluing together copies of $T$. Certain boundary edges of $\mathcal{F}_n$ will then be equivalent to each other. When these boundary edges are identified, the region $\mathcal{F}_n$ becomes a Riemann surface. We will denote this surface by $\Gamma_n/\mathcal{H}$. These surfaces, and their associated combinatorial objects, have received considerable attention in the literature, most notably in [2], [3] and [4].

In [2], Brooks, Perry and Petersen used Cayley graphs of $PSL(2, \mathbb{Z}_n)$ to construct the Riemann surfaces $\Gamma_n/\mathcal{H}$. Specifically, they set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$ 

The set $S$ is a symmetric generating set for $PSL(2, \mathbb{Z}_n)$. The Cayley graph for $PSL(2, \mathbb{Z}_n)$ with respect to the generating set $S$ is the graph with one vertex for each group element, with $g_1, g_2$ connected if $g_1 = sg_2$ for some $s \in S$. They then took one copy of $T$ for each vertex, and glued boundary edges together according to the edges of the graph. By constructing the surfaces in this way, they were able to use combinatorial properties of the graphs to answer geometric questions about the surfaces.

In [2], the authors used Cayley graphs of matrix groups to construct arithmetic Riemann surfaces. In this note we show that this perspective can be reversed. In other words, tesselations of arithmetic Riemann surfaces can be used to obtain Cayley graphs of $PSL(2, \mathbb{Z}_n)$. As a tool for doing that, we will also construct an explicit action of the group $PSL(2, \mathbb{Z}_n)$ on the surfaces $\Gamma_n/\mathcal{H}$.

We associate a graph $G_n$ to $\Gamma_n/\mathcal{H}$ in the following way: There is one vertex for each copy of $T$ in $\Gamma_n/\mathcal{H}$. Two vertices are connected if they represent triangles sharing a boundary edge. Since the region of $\mathcal{H}$ inducing $\mathcal{F}_n$ is simply connected, we can be sure that $G_n$ is a cubic graph. Our main theorem is the following:

**Theorem 1.** Let $n > 2$ be a positive integer. Then the graph $G_n$ is isomorphic to the Cayley graph for $PSL(2, \mathbb{Z}_n)$ with respect to the generating set $S$.

Our definition of $T$ above implies that it is adjacent to those triangles obtained by applying to it the fractional linear transformations $z \mapsto z + 1$, \begin{align*}
S &= \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.
\end{align*}
$z \mapsto z - 1$, and $z \mapsto -\frac{1}{z}$. These are precisely the functions induced by the elements of $S$. We will prove the main theorem by showing that the automorphism group of the graphs $G_n$, denoted by $Aut(G_n)$, contains a subgroup isomorphic to $PSL(2, \mathbb{Z})$ that acts regularly on the vertices of $G_n$ (by which we mean that within this subgroup the only automorphism with fixed points is the identity.) This will imply that $G_n$ is a Cayley graph for $PSL(2, \mathbb{Z})$ (see [1]).

2. Proof of Theorem One

In this section we define an explicit action of the group $PSL(2, \mathbb{Z}_n)$ on the Riemann surface $\Gamma_n/\mathcal{H}$.

Set $K = SO(2, \mathbb{R})$ (i.e. the group of $2 \times 2$, orthogonal, real matrices with determinant one). Set $G = SL(2, \mathbb{R})$. The Bruhat decomposition for $\mathcal{H}$ tells us that $\mathcal{H} \cong G/K$. The isomorphism is given explicitly by mapping the coset $gK$ to $g(i)$, the image of $i$ under the fractional linear transformation induced by $g$ (see [6]).

It follows from this that the space $\Gamma_n/\mathcal{H}$ can be viewed as the double coset space $\Gamma_n \backslash G/K$. Recall that since $\Gamma / \Gamma_n \cong PSL(2, \mathbb{Z}_n)$, we can view elements of $PSL(2, \mathbb{Z}_n)$ as cosets $g\Gamma_n$ where $g \in \Gamma$.

We now define our action. Let $g \in \Gamma$ and let $g\Gamma_n \in \Gamma / \Gamma_n$. Choose $\gamma \in SL(2, \mathbb{R})$ and let $\Gamma_n \gamma K \in \Gamma_n / \mathcal{H}$. Then we define the function $\Phi_g : \Gamma_n / \mathcal{H} \rightarrow \Gamma_n / \mathcal{H}$ by

$$\Phi_g(\Gamma_n \gamma K) = \Gamma_n g^{-1} \gamma K.$$ 

This definition required two choices of coset representatives. The following propositions establish that this function is well-defined.

**Proposition 1.** Let $g, h \in \Gamma$ be representatives of the same coset in $\Gamma / \Gamma_n$. Then $\Phi_g = \Phi_h$. Further, if $\gamma, \gamma'$ represent the same coset in $G/K$, then $\Phi_g(\Gamma_n \gamma K) = \Phi_g(\Gamma_n \gamma' K)$.

**Proof.** Let $g, h$ be as above, and let $\Gamma_n \gamma K \in \Gamma_n / \mathcal{H}$. We must show that

$$\Gamma_n g^{-1} \gamma K = \Gamma_n h^{-1} \gamma K.$$ 

This is equivalent to showing that $g^{-1} \gamma$ and $h^{-1} \gamma$ are representatives of the same double coset in $\Gamma_n / \mathcal{H}$. This will be true precisely when there is a fractional linear transformation mapping $g^{-1} \gamma K$ to $h^{-1} \gamma K$. This is equivalent to showing that $g \gamma K$ and $h \gamma K$ are equivalent under the action of $\Gamma$.

Since $g, h$ represent the same coset in $\Gamma / \Gamma_n$, we know that $g^{-1} h \in \Gamma_n$. It follows that there is an $M \in \Gamma_n$ such that $M g = h$. So we have $h \gamma K = (Mg) \gamma K = g \gamma K$, and the proof of the first part of the proposition is complete.
For the second part, notice that since $\gamma, \gamma'$ represent the same coset in $G/K$, we have that $\gamma K = \gamma' K$. Since this implies $g\gamma K = g\gamma' K$, the proof is complete. \[\square\]

This establishes that $\Phi_g$ is well-defined. The next two propositions show that $\Phi_g$ is a homomorphism and that it is continuous.

**Proposition 2.** Let $g, h \in \Gamma$. Then $\Phi_g \circ \Phi_h = \Phi_{gh}$.

**Proof.** Let $g, h$ be as above and let $\Gamma_n \gamma K \in \mathcal{H}/\Gamma_n$. Then we compute:

$$\Phi_g(\Phi_h(\Gamma_n \gamma K)) = \Phi_g(\Gamma_n h^{-1} \gamma K)$$

$$= \Gamma_n h^{-1} g^{-1} \gamma K$$

$$= \Gamma_n (gh)^{-1} \gamma K$$

$$= \Phi_{gh}(\Gamma_n \gamma K),$$

as desired. \[\square\]

**Proposition 3.** The function $\Phi_g : \Gamma_n \backslash G/K \to \Gamma_n \backslash G/K$ is continuous.

**Proof.** The commutative diagram below will help to follow the argument.

\[
\begin{array}{ccc}
\mathcal{H} = G/K & \xrightarrow{\Phi'_\gamma} & \mathcal{H} = G/K \\
\pi^{-1} & \downarrow & \downarrow \pi \\
\Gamma_n \backslash G/K & \xrightarrow{\Phi_g} & \Gamma_n \backslash G/K
\end{array}
\]

It is trivial to show that $\Phi_\gamma = \pi \Phi'_\gamma \pi^{-1}$. But since $\Gamma_n \backslash G/K$ is equipped with the quotient topology from $G/K$, we know that $\pi$ and $\pi^{-1}$ are continuous functions. It only remains to show that $\Phi'_\gamma$ is continuous as well.

Under the Bruhat decomposition, the coset $\gamma g K$ corresponds to the point $z = \gamma(g(i)) = \gamma g(i)$. But now the continuity of $\Phi'_\gamma$ follows from the continuity of fractional linear transformations, and the proof is complete. \[\square\]

Finally, we show that $\Phi_g$ acts transitively on the triangles of $\Gamma/\mathcal{H}$.

**Proposition 4.** If $T$ is a modular triangle, then so is $\Phi_g(T)$. Furthermore, if $T_1, T_2$ are modular triangles, then there is a unique $g \in \text{PSL}(2, \mathbb{Z}_n)$ such that $\Phi_g(T_1) = T_2$.

**Proof.** Fractional linear transformations preserve geodesics and angles. This implies that $\Phi_g$ preserves modular triangles. For the second part of the proposition, note that no two elements of $\text{PSL}(2, \mathbb{Z}_n)$ induce the same fractional linear transformation. Since there are $\mu$ distinct, triangle-preserving
transformations and $\mu$ many triangles in $\Gamma_n/\mathcal{H}$, the second part of the proposition is proved.

We can now prove the main theorem.

Proof. Main Theorem. Since the functions $\Phi_g$ are continuous and preserve triangles, it is clear that they induce automorphisms of the graphs $G_n$. If $g, h \in PSL(2, Z_n)$ are distinct, then we know proposition four that $\Phi_g \neq \Phi_h$. This implies that $PSL(2, Z_n)$ is a subgroup of $Aut(G_n)$. Finally, it is clear from our definition of $\Phi_g$ that if $g$ is not the identity then $\Phi_g$ does not fix any triangles. This implies that $PSL(2, Z_n)$ acts regularly on $G_n$. It follows from this that $G_n$ is a Cayley graph of $PSL(2, Z_n)$. The appropriate generating set is obtained by determining the elements of $PSL(2, Z_n)$ mapping an arbitrary vertex in $G_n$ to its neighbors. It is clear from our geometric definition of the graphs $G_n$ that $S$ is the appropriate generating set, and the proof is complete.

References


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