Find a four-digit number whose digits reverse upon being multiplied by four.

SOLUTION: The answer is 2178.

Note that the first digit of our number must be 2. If it were 3 or higher, then our number, when multiplied by 4, would give a result that is greater than 10000. If it were 1, then the units digit of our number, when multiplied by four, would have to produce a units digits of 1, which is impossible.

It follows that the units digit of our product must be 2 (since we are given that the digits reverse). That means the units digit of our original number would have to be either 3 or 8. However, 3 is impossible. To see this, suppose our original four-digit number is $2ab3$, where a and b are the two unknown digits in our number. Since the digits reverse when multiplied by four, we obtain the equation

$$4(2000 + 100a + 10b + 3) = 3000 + 100b + 10a + 2$$
$$8000 + 400a + 40b + 12 = 3000 + 100b + 10a + 2$$
$$5000 + 390a + 10 = 60b.$$

This last equation is plainly impossible. Even if we make the left-hand side as small as possible by setting $a = 0$, and the right-hand side as large as possible by setting $b = 9$, the right-hand side is still way too small.

It follows that our units digit is 8, and our four-digit number has the form $2ab8$. Now we can rewrite the above equations to obtain

$$4(2000 + 100a + 10b + 8) = 8000 + 100b + 10a + 2$$
$$8000 + 400a + 40b + 32 = 8000 + 100b + 10a + 2$$
$$390a + 30 = 60b$$
$$195a + 15 = 30b.$$

It is clear that b must be larger than a. So, by trial and error starting with $b = 9$, we quickly hit upon $b = 7$ and $a = 1$ as the only possible solution. So the answer is 2178, as claimed.