Problem of the Week
Solution Six

In the addition problem below, each letter stands for a different digit. However, each letter stands for the same digit in every place where it appears. Find the only possible value for each digit to make a correct addition statement:

$$\begin{array}{cccccccc} & & D & O & U & B & L & E \\ & & D & O & U & B & L & E \\ + & & & & T & O & I & L \\ \hline & T & R & O & U & B & L & E \end{array}$$

The solution is: 798,064 + 798,064 + 1,936 = 1,598,064

The hardest part in a problem like this is finding a place to begin. In this case, the key observation is that we can say at once that $O = 9$. Simply by running through all the cases, we can see that the only way to add two digits to get a sum whose ones digit is what you started with is by taking $9 + 9 + 1$, where the 1 represents a carry from the previous column.

It now immediately follows that $T = 1$. This follows because D is no larger than 8, and it is not possible to add two six-digit numbers smaller than $900{,}000$ to a four-digit number and produce a sum larger than $2{,}000{,}000$. So, we have now worked out two letters, leaving us with this:

$$\begin{array}{cccccccc} & & D & 9 & U & B & L & E \\ & & D & 9 & U & B & L & E \\ + & & & & 1 & 9 & I & L \\ \hline & 1 & R & 9 & U & B & L & E \end{array}$$

At this point we can quickly determine both B and U. We cannot have $B = 1$ (trying to use the fact that $1 + 1 + 9 = 11$), since 1 has already been used. (This possibility is also ruled out by the inevitability of a carry from the previous column, but that is a fancier argument than we need.) By quickly running through the possibilities we find that the only other possibility is that $B = 0$. With the carry from the previous column we have $0 + 0 + 9 + 1 = 10$. This produces a carry into the next column, and it is once more a matter of running through the possibilities to see that $U = 8$.

Our problem has now been reduced to this:

$$
\begin{array}{cccccccc}
 & & D & 9 & 8 & 0 & L & E \\
 & & D & 9 & 8 & 0 & L & E \\
+ & & & & 1 & 9 & I & L \\
\hline
 & 1 & R & 9 & 8 & 0 & L & E
\end{array}
$$

Our next observation is that D is either $5, 6, 7, 8$, or 9. This follows from the need for a carry into the next column. But we cannot have $D = 8$ or $D = 9$, since those digits have been used. Nor can we have $D = 5$, since that would force us to have $R = 0$, but that digit has also been used. We conclude that $D = 6$ or $D = 7$.

Time for the big finale. Look at the ones column. One possibility is that $E = 3$ and $L = 7$, taking advantage of the fact that $3 + 3 + 7 = 13$. In the tens column we would then find that $I = 2$, since we would need to use the fact that $7 + 7 + 2 + 1 = 17$ (keeping in mind the carry from the previous column). But this is not possible! For now, since 7 is taken, we would have to have that $D = 6$. This forces $R = 2$, which means that 2 would be used twice. Of course, the possibility that $E = 7$ and $L = 3$ is ruled out by the same argument.

It follows that we have either that $E = 4$ and $L = 6$, or $E = 6$ and $L = 4$. Trying both in turn quickly reveals that only the former works. This leads to the solution

$$
\begin{array}{cccccccc}
 & & 7 & 9 & 8 & 0 & 6 & 4 \\
 & & 7 & 9 & 8 & 0 & 6 & 4 \\
+ & & & & 1 & 9 & 3 & 6 \\
\hline
 & 1 & 5 & 9 & 8 & 0 & 6 & 4
\end{array}
$$