
Problem of the Week

Solution Eight

Using the digits from 1–9 exactly once each, what are the two numbers which multiply to give the largest possible product?

SOLUTION: The maximum product occurs by taking the two numbers to be 9642 and 87531.

It is clear that given two numbers whose digits are not in decreasing order, a greater product will occur by rearranging the digits in decreasing order from left to right. Such a rearrangement can no cause either number to become smaller, while at least one of the numbers will plainly become larger.

It is also clear that in our problem the maximum product will occur from using a five-digit number and a four-digit number. This is somewhat tedious to prove rigorously, but the idea is this: Suppose you tried a six-digit number and a three-digit number. Then a greater product can be obtained by deleting the units digit of the six-digit number and inserting it into the correct place (pursuant to the requirement that the digits appear in decreasing order.)

Finally, there is a basic principle that says that among all pairs of numbers with a given sum, the product increases as the difference between the two numbers decreases. Thus, it is now clear how we should build our numbers. We want a five-digit number and a four-digit number whose digits are in decreasing order and whose difference is as small as possible. Thus, we would start with the pair 9 and 8. Then we would obtain 96 and 87. This represents the largest product that can be obtained from using just four distinct digits. Then we obtain 964 and 875, and 9642 and 8753. Finally, the maximum product occurs by placing the 1 at the end of the smaller number, giving us the answer 9642 and 87531.