Problem of the Week
Solution Nine

Have a go at this one:

Using the digits from 1–9 exactly once each, form three, three-digit numbers such that one of them is the sum of the other two.

Here are a few possible answers:

- 475+218=693
- 357+291=648
- 273+186=459

There are many others.

In finding these solutions, some amount of trial and error is inevitable. However, with some cleverness you can narrow down the possibilities. When I posted this problem to my blog, a commenter calling himself Sean T offered the following clever argument for showing that the sum of the digits in the answer must be 18, and that there must be exactly one carry in the addition problem:

"I was not able to get completely to solutions analytically, but I was able to get partially there. We know that the sum of the digits from 1–9 is 45. Let $ABC + DEF = GHI$. Further, let $x = A+B+C+D+E+F$ and $y = G+H+I$. Now, if there are no carries, it is clear that x=y. Since $x + y = 45$ and $x - y = 0$, adding those two equations gives $2x = 45$ or $x = 22.5$, which is impossible since x is the sum of integers. Thus, there must be at least one carry.

If there are two carries, then they must be in the ones and tens places. Thus $C + F = 10 + I$, $1 + B + E = 10 + H$, and $1 + A + D = G$. Thus $2 + x = 20 + y$, yielding $x - y = 18$. Since $x + y = 45$, adding the last two equations gives $2x = 27$ or $x = 13.5$, which again is impossible since x is the sum of integers.

This implies that there must be exactly one carry. This can be in either the ones or the tens place. For the following, I will assume the carry is in the ones place. It will not matter for my conclusion whether it is in the ones or the tens place. Now, a carry in the ones place implies that $C+F = 10+I$, $1 + B + E = H$ and $A + D = G$. This yields $x + 1 = 10 + y$ or $x - y = 9$. Adding that to the equation $x + y = 45$ gives $2x = 54$ or $x = 27$. That implies that $y = 18$. Therefore, the solutions to the problem will all have the sum of the three digits in the answer equal to 18.

It is fairly easy to generate solutions from that point by trial and error. For instance $9+7+2 = 18$ so there should be a solution with those three digits as the answer. We know we need one carry, so select $8 + 4 = 12$ to account for the two in the answer. $5 + 1$ plus the carry gives the 7 and the remaining digits $6 + 3 = 9$ so one solution would be $658 + 314 = 972$. Obviously, as well noted above, other solutions can be generated from this one by swapping digits in the addends."