Problem of the Week
Number Two
September 14, 2015

In the 1998 film *Good Will Hunting*, there's a scene where math genius Matt Damon and MIT math professor Stellan Skarsgård are staring at a problem on a chalkboard. Damon makes a gesture indicating the problem is simple. He approaches the board, chalk in hand, and taps on the board in a few places. Skarsgård smiles, indicating that he likewise sees the solution. Soon they are both writing furiously on the board. The problem having been solved, they proceed to high five each other. End scene.

If you watch this scene very carefully–and who hasn't?–you can actually make out what they are doing at the board. It turns out they are just reducing a fraction. The writing involved crossing out common terms on the top and bottom.

Now, mathematicians rarely get so excited about reducing fractions, and we even more rarely high five each other in any event. It is true, though, that people have some strange ideas about fraction reduction. Consider a fraction like $\frac{a+b}{a+c}$. Beginning algebra students, seeing only alphabet soup, routinely "simplify" this to $\frac{b}{c}$, by canceling out the a's. That this is erroneous can be seen by plugging in almost any numbers you like for a, b, and c.

We can have some fun with this sort of thing. Once you have moved on to trigonometry you might be inclined to say

$$\frac{\sin x}{n} = 6.$$

You know, by canceling out the n's. Then again, sometimes this sort of thing works very well:

$$\frac{16}{64} \qquad \frac{26}{65} \qquad \frac{19}{95} \qquad \frac{49}{98}$$

In each case, canceling out the common digit on top and bottom leads to a correct result.

These are the only two-digit fractions where this works, excepting the trivial cases such as $\frac{11}{11}$ or $\frac{22}{22}$. There are three-digit examples, though:

$$\frac{484}{847} = \frac{4}{7} \qquad \frac{545}{654} = \frac{5}{6}$$

There are four and five-digit examples as well:

$$\frac{3243}{4324} = \frac{3}{4} \qquad \frac{14714}{71468} = \frac{14}{68} = \frac{7}{34}.$$

And a six-digit example:

$$\frac{878048}{987804} = \frac{8}{9}$$

You can take this to absurd extremes:

$$\frac{12345679}{98765432} = \frac{1}{8}.$$

Yes, really. Just cancel out all of the common digits on top and bottom, and only the 1 on top and the 8 on the bottom survive.

What does this have to do with this week's problem? Not much, really, except that our false proof this week also features a fraction.

Remember how this works. I'm going to present an argument with an obviously false conclusion. You must explain, *with crystal clarity*, where the argument goes wrong. Are you ready? Here we go $\Longrightarrow$

I'm going to prove that $7 = 13$. Let x denote a solution to the equation

$$\frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}.$$

By finding a common denominator on the left-hand side we obtain

$$\frac{x+5-5(x-7)}{x-7} = \frac{4x-40}{13-x}.$$

Simplifying the left-hand side leads to

$$\frac{-4x+40}{x-7} = \frac{4x-40}{13-x}.$$

This implies that

$$\frac{4x-40}{7-x} = \frac{4x-40}{13-x}.$$

Now canceling the $4x - 40$ on both sides and reciprocating what remains leads to $7 - x = 13 - x$. This implies $7 = 13$ as claimed.

When you think you've spotted the error, follow these instructions:

*Submissions are due to Jason Rosenhouse by 5:00 on **Friday, September 18**. Solutions should be written on the back of an official POTW handout. Place your name, e-mail address, and the section numbers and professors of any math courses you are taking, in the **upper right corner** of the front of the page. One weekly winner will receive a five-dollar gift card from Starbucks. Answers will be judged on the clarity with which they explain the flaw in the argument. Solutions will be posted at this website, by the Monday after the problem is due:*

http://educ.jmu.edu/∼rosenhjd/POTW/Fall15.html