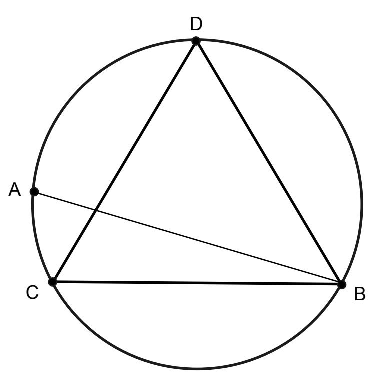
Problem of the Week
Number Four
September 28, 2015

This week's problem involves trigonometry, and trigonometry is about triangles. So how about we warm-up with a classic problem involving tricky, triangle-based arguments?

This is an old conundrum due to Joseph Betrand. In his 1889 book *Calculus of Probabilities*, he posed the following problem: Given a randomly chosen chord in a fixed circle, what is the probability that the chord is longer than the side of the equilateral triangle inscribed in the circle? (Keep in mind that any two equilateral triangles inscribed in the same circle must have the same side length.)

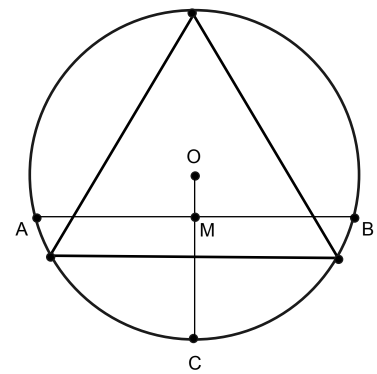
We shall consider three possible solutions.

METHOD ONE: Let A and B denote the endpoints of a randomly chosen chord. Rotating if necessary, let one of the vertices of the equilateral triangle coincide with B, as shown in our first diagram:



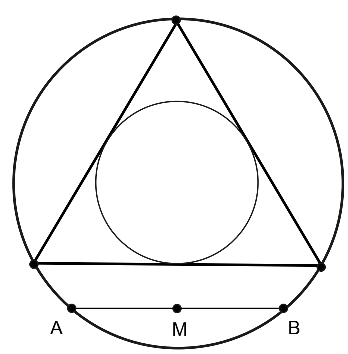
The chord will be longer than a side of the triangle precisely when A lies on the shorter arc cut off by C and D. Since this arc accounts for one-third of the circle, the probability we seek is $\frac{1}{3}$.

METHOD TWO: This time we choose an arbitrary radius, represented by OC in the second diagram. We choose an arbitrary point M on this radius, and then construct the unique chord, represented by AB, perpendicular to the radius at the point M. In this way we have selected a random chord. Now rotate the equilateral triangle until one of its sides is perpendicular to radius OC:



Standard results in geometry now tell us that this side of the triangle bisects OC, and also that this side is parallel to AB. It is clear that in the lower semicircle, the chord is longer than the side precisely when M is inside the triangle, which happens with probability $\frac{1}{2}$. By symmetry, this applies to the upper semicircle as well. We conclude that $\frac{1}{2}$ is the probability we seek.

METHOD THREE: This time we choose an arbitrary point M inside the circle. We now let AB denote any chord of the circle having M as its midpoint. Rotate the triangle so that one of its sides is parallel to AB. Finally, inscribe a circle inside the triangle. The result is shown in the third diagram.



Now, the chords that are longer than a side of the triangle are precisely those whose midpoints are within the inscribed circle. (This is non-trivial, but it makes sense if you stare at the picture for a while.) Standard results in geometry tell us that the area of the inscribed circle is one-fourth of the large circle. So the probability we seek is $\frac{1}{4}$.

So there you go. Three plausible arguments leading to three different answers. Which is correct? You can mull that over, but we shall defer further discussion to the solutions.

After all that, the actual POTW is likely to seem anti-climactic. But let's have a look anyway. Remember, your task is to pinpoint the error in the following argument.

I will now prove that the cosine of any acute angle is greater than one. Let α be an acute angle. Then $\cos\alpha > 0$. It is clear that

$$\ln\cos\alpha = \ln\cos\alpha.$$

By doubling the left-hand side we obtain the inequality

$$2\ln\cos\alpha > \ln\cos\alpha.$$

A standard property of logarithms now gives us

$$\ln\cos^2\alpha > \ln\cos\alpha.$$

Since the natural logarithm function is strictly increasing, we conclude that

$$\cos^2\alpha > \cos\alpha.$$

If we now divide both sides by $\cos\alpha$, we are left with $\cos\alpha > 1$, as claimed.

Got that? Either rewrite the rules of trigonometry or explain to me what went wrong *this* time. Your choice.

Submissions are due to Jason Rosenhouse by 5:00 on ***Friday, October 2****. Solutions should be written on the back of an official POTW handout. Place your name, e-mail address, and the section numbers and professors of any math courses you are taking, in the* ***upper right corner*** *of the front of the page. One weekly winner will receive a five-dollar gift card from Starbucks. Answers will be judged on the clarity with which they explain the flaw in the argument. Solutions will be posted at this website, by the Monday after the problem is due:*

http://educ.jmu.edu/∼rosenhjd/POTW/Fall15.html