Problem of the Week Number Seven October 26, 2015

With this week's problem we have our first "false proof" arising from calculus. A major topic in calculus is the proper handling of infinite series. This is important, since an improper handling of infinite series can lead to things like this:

Let us define S to be the value of the sum

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots.$$

Multiplying both sides by two gives us

$$2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \dots$$

But this simplifies to

$$2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots$$

If this is not clear, simply write out more of the terms of $2S$, group together terms with the same denominator, and then watch the pattern emerge.
But this implies that $2S = S$. Since S is plainly not zero we can divide by it to obtain $2 = 1$.

Strange! And it becomes even stranger when you consider that S is the alternating harmonic series, which is well-known to converge to $\ln 2$. The problem with our argument is that the commutative property of addition does not hold for conditionally convergent series, which S and $2S$ both are. If you rearrange the terms of such a series, you might get a different answer. And we arranged the series to make $2S$ look like S.

This observation leads to a very counter-intuitive theorem in analysis: A conditionally convergent series can be rearranged to add up to any real number you like. That sounds like the end result of an argument I would use for POTW this term, but it is actually correct. The basic idea is not hard to sketch.

Let S be a conditionally convergent series. Separate out the positive and negative terms. It is readily shown that there are infinitely many of both, and that each represents a divergent series by itself. Now pick a number, say π. To make the series sum to π, keep taking positive terms until the sum is greater than π. Then take negative terms until the sum is smaller that π. Then go back to the positive terms until the sum is greater than π. Then more negative terms until it is smaller. Since the sequences of positive terms and negative terms are both going to zero (the whole series is conditionally convergent, recall) we find that the amount of overshoot or undershoot must be going to zero as well. And there you go!

In math, it sometimes happens that what initially look like nonsense turns out to be true. Alas, more often what looks like nonsense is nonsense, and that is the case with this week's false proof. Here it is:

I am going to prove that $2 = 1$. Let x represent a positive number. We begin with the observation that $x^2 = x(x)$. We can expand the right-hand side to obtain

$$x^2 = x\,\underbrace{(1 + 1 + \cdots + 1)}_{x \text{ times}}.$$

Distributing on the right-hand side gives us

$$x^2 = \underbrace{x + x + \cdots + x}_{x \text{ times}}.$$

(For example, we can write $3^2 = 3(3) = 3(1 + 1 + 1) = 3 + 3 + 3$.) But now we can take derivatives on both sides to obtain

$$2x = \underbrace{(1 + 1 + \cdots + 1)}_{x \text{ times}}.$$

This implies that $2x = x$. Since x was assumed to be positive, we can divide by it to conclude that $2 = 1$

When you have figured out what went wrong, let me hear about it! But please follow these instructions:

Submissions are due to Jason Rosenhouse by 5:00 on ***Friday, October 30****. Solutions should be written on the back of an official POTW handout. Place your name, e-mail address, and the section numbers and professors of any math courses you are taking, in the* ***upper right corner*** *of the front of the page. One weekly winner will receive a five-dollar gift card from Starbucks. Answers will be judged on the clarity with which they explain the flaw in the argument. Solutions will be posted at this website, by the Monday after the problem is due:*

http://educ.jmu.edu/∼rosenhjd/POTW/Fall15.html