# Problem of the Week
# Number Eight
# November 2, 2015

Last week we gave an example of the mayhem that ensues when you are too casual about manipulating infinite series. Let me show you an even more shocking example of what can happen.

We start with the series

$$S_1 = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

To assign a numerical value to $S_1$, we notice that if we cut off the series after an odd number of terms we get a sum of $1$, while if we cut it off after an even number of terms we get a sum of $0$. So the only sensible thing is to take the average and declare that $S_1 = \frac{1}{2}$.

Stop laughing and just stay with me for a moment!

Now define the series

$$S_2 = 1 - 2 + 3 - 4 + 5 - \dots$$

To evaluate this, let us add two copies of the series, with the second copy shifted by one, like this:

$$\begin{array}{ccccccccccc} 1 & - & 2 & + & 3 & - & 4 & + & 5 & \dots & \\ & & 1 & - & 2 & + & 3 & - & 4 & \dots & \end{array}$$

The conclusion is that

$$2S_2 = 1 - 1 + 1 - 1 + 1 - \dots = S_1 = \frac{1}{2}.$$

Plainly, then, we have that $S_2 = \frac{1}{4}$.

We now define our final series like this:

$$S = 1 + 2 + 3 + 4 + 5 + \dots$$

Evaluate $S - S_2$ as follows:

$$\begin{aligned} S - S_2 &= (1 + 2 + 3 + 4 + \dots) - \\ &\qquad (1 - 2 + 3 - 4 + \dots) \\ &= (1 - 1) + (2 - (-2)) + \\ &\qquad (3 - 3) + (4 - (-4)) + \dots \\ &= 4 + 8 + 12 + 16 + \dots \\ &= 4(1 + 2 + 3 + 4 + \dots) \\ &= 4S. \end{aligned}$$

Since we know that $S_2 = \frac{1}{4}$, we can substitute and solve for $S$. We get the equation

$$S - \frac{1}{4} = 4S,$$

which is easily solved to find that $S = -\frac{1}{12}$.

Did you catch that? We just proved

$$1 + 2 + 3 + 4 + 5 + \dots = \frac{-1}{12}.$$

This result was actually first noticed by Euler. The sum of infinitely many positive whole numbers is a negative fraction. Really?

Now, if you think about infinite series the way we teach you in second-semester calculus, then this should just seem absurd. These are divergent series and that's all there is to it. You can't just naively do algebra with divergent series and expect to get a reasonable result. Done and done.

Maybe. But now consider this: There's something called the Riemann zeta-function, which is defined like this:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

If you think of $s$ as being a positive real number, then we are solidly in second-semester calculus territory. Ye olde ratio test will show you that it converges if $s > 1$ and will diverge otherwise. With just a bit more work you can assume $s$ is a complex number, and when you do you will find that the series only converges when the real part of $s$ is greater than one.

Now, here's the thing. There is another, unique, function which we shall call $Z(s)$. It is defined and differentiable everywhere in the plane (with the exception of a single point (s=1) which need not concern us), and it has this strange property that it will give you exactly the same answer as $\zeta(s)$ if you only use values of $s$ whose real part is greater than one. In the math biz we say that $Z(s)$ is an "analytic continuation" of $\zeta(s)$.

If we formally plug $s = -1$ into zeta, ignoring the fact that we get a divergent series, then we find that

$$\zeta(-1) = 1 + 2 + 3 + 4 + 5 + \dots$$

And it turns out that $Z(-1) = \frac{-1}{12}$. Really.

Are you convinced now? Have we shown that the sum of all the positive integers really is $\frac{-1}{12}$? We should keep in mind that these are infinite sums we are talking about. That means it is meaningless to talk about what they *actually* equal. We can only describe different conventions for handling them and judge them based on their usefulness.

Is this where you pounce? Do you reply, "You can make whatever strange definitions you want, and if it amuses you to assign a value to the sum of the integers based on the analytic continuation of the zeta function then you're welcome to do so. But we have certainly left behind any intuition we have ever had about addition, and your convention is not useful."

But this is where the story gets truly bizarre. You see, sums like $1 + 2 + 3 + \dots$ arise naturally in a number of physical applications. Physicists have discovered that they sometimes get empirically meaningful results by declaring that series to be equal to $\frac{-1}{12}$. Yes, really. Make of that what you will.

The moral of the story is this: Sometimes it has practical value to investigate "obviously" absurd arguments! Kind of justifies this whole semester of POTW, don't you think? Not to mention the fact that it's fun. So have a go at this week's false proof, and when you think you've spotted the error let me know.

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I shall prove that $1 = 0$. To do that, I will use integration by parts to evaluate $\int \frac{1}{x}\, dx$. Make the following definitions:

$$u = \frac{1}{x} \qquad dv = dx$$

$$du = -\frac{1}{x^2} \qquad v = x$$

Recall that the formula for integration by parts is this:

$$\int u\, dv = uv - \int v\, du.$$

Applying this formula to the present case gives us:

$$\int \frac{1}{x}\, dx = \left(\frac{1}{x}\right) x - \int x \left(\frac{-1}{x^2}\right),$$

which simplifies to

$$1 + \int \frac{1}{x}\, dx.$$

Therefore, we have discovered that

$$\int \frac{1}{x}\, dx = 1 + \int \frac{1}{x}\, dx.$$

Subtracting the integral from both sides leads to the conclusion that $0 = 1$, as claimed.

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*Submissions are due to Jason Rosenhouse by 5:00 on* ***Friday, November 6****. Solutions should be written on an official POTW handout. Place your name, e-mail address, and the section numbers and professors of any math courses you are taking, in the* ***upper right corner*** *of the front of the page. One weekly winner will receive a five-dollar gift card from Starbucks. Answers will be judged on the clarity with which they explain the flaw in the argument. Solutions will be posted at this website, by the Monday after the problem is due:*

**http://educ.jmu.edu/~rosenhjd/POTW/Fall15.html**