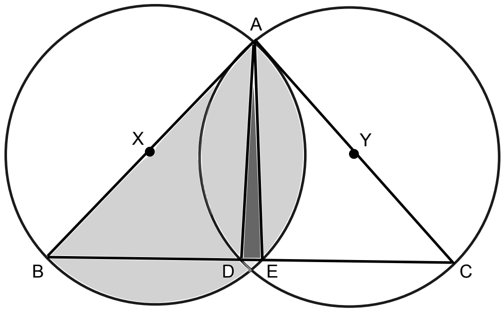
Problem of the Week Solution Three

PROBLEM: I'm going to show that it is possible to construct a triangle with two right angles:

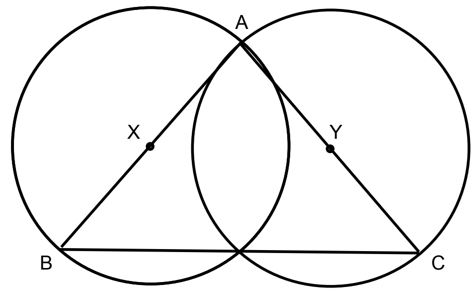


We have two overlapping circles. One has its center at X, while the other has its center at Y. We draw diameters AB and AC. Then we connect points B and C. This line segment intersects one circle at point D and the other at point E. The result is the tall, thin triangle ADE, shaded dark grey in the figure.

Now notice that angle AEB (which is the same as angle AED) is inscribed in semicircle $AEBX$, shaded light grey in the figure. It is a standard theorem in geometry that an angle inscribed in a semicircle is a right angle. So angle AEB is a right angle, and therefore so is angle AED. Likewise, angle ADC is inscribed in semicircle $ADCY$. Therefore, ADE is a right angle as well. It follows that AED is a triangle with two right angles, which shows that such a thing is possible.

SOLUTION: Math teachers will often say, "A picture is never a proof!" Drawing a clear diagram is often essential to finding a proof, but it is not a proof by itself. This problem shows why.

The diagram is subtly misleading. The points X and Y are not perfectly in the centers of their circles, which means that AB and AC are not actually diameters. If they were, the diagram would have looked like this:



Notice that line segment BC passes through the second point of intersection of the two circles. That means that D and E are actually the same point, and therefore it is not possible to draw the dark grey triangle of the original diagram.

In fact, what we have here is essentially a proof by contradiction of the following theorem: Given two overlapping circles, the line segment joining the endpoints of the diameters drawn from one point of intersection of the circles passes through the second point of intersection of the circles.

That's a bit of a mouthful, but I think the diagram makes it clear what I have in mind. Technically we also need to consider the possibility that line segment BC is not entirely contained within the two circles, but that can be shown to be impossible.