Problem of the Week
Solution Four

PROBLEM: Given a randomly chosen chord in a fixed circle, what is the probability that the chord is longer than the side of the equilateral triangle inscribed in the circle? (Keep in mind that any two equilateral triangles inscribed in the same circle must have the same side length.)

SOLUTION: We gave three plausible arguments that led to three different answers. None of the arguments, however, is wrong! The trouble arises because the notion of "choosing a random chord" is ambiguous. Our three arguments employed different interpretations of this notion. Each leads to a different probability distribution on the set of chords, and that is why each led to a different answer. The problem cannot be definitively solved until a clear procedure for choosing the chord is specified. A more detailed discussion of this problem can be found in the Wikipedia article, "Bertrand paradox (probability)."

MAIN PROBLEM: I will now prove that the cosine of any acute angle is greater than one. Let α be an acute angle. Then $\cos\alpha > 0$. It is clear that

$$\ln\cos\alpha = \ln\cos\alpha.$$

By doubling the left-hand side we obtain the inequality

$$2\ln\cos\alpha > \ln\cos\alpha.$$

Standard properties of logarithms now gives us

$$\ln\cos^2\alpha > \ln\cos\alpha.$$

Since the natural logarithm function is strictly increasing, we conclude that

$$\cos^2\alpha > \cos\alpha.$$

If we now divide both sides by $\cos\alpha$, we are left with $\cos\alpha > 1$, as claimed.

SOLUTION: The error is in the first inequality. Since $0 < \cos\alpha < 1$, we know that $\ln\cos\alpha$ is negative. When a negative number is doubled, it becomes smaller. Therefore, the correct inequality is

$$2\ln\cos\alpha < \ln\cos\alpha.$$

The rest of the argument is correct, but it will now lead to the conclusion that $\cos\alpha < 1$, which we already knew.