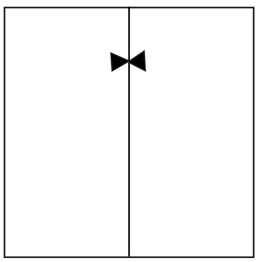
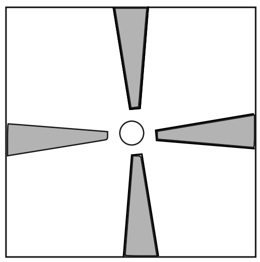
Problem of the Week
Solution Five

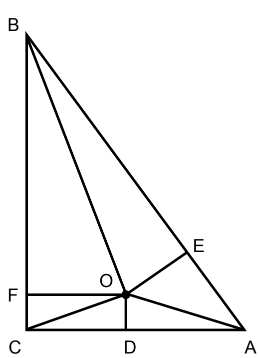
PROBLEM: What do you make of these Droodles?





SOLUTION: Clearly, the first is a man in a tuxedo who stood too close to the elevator doors, while the second is four elephants checking out a tennis ball.

PROBLEM: I will now prove that the hypotenuse of a right triangle is equal to one of its legs. We start with triangle ABC, with right angle at C.



We draw the angle bisector from B and the perpendicular bisector of AC, and we denote by O the point where they intersect. (Thus, we have that $\angle CBO = \angle ABO$, $CD = AD$, and $\angle ODA = \angle ODC = 90$.) We also draw segment OF perpendicular to BC and segment OE perpendicular to AB.

We now observe that $\triangle BOE \cong \triangle BOF$. Both are right triangles, they have a common side, and their angles at B are equal since OB is an angle bisector. It follows that $BE = BF$.

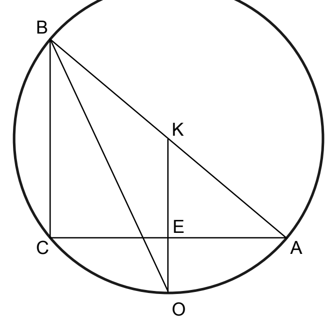
We also know that $\triangle OFC \cong \triangle OEA$. To see this, notice first that both are right triangles. Then note that since OD is a perpendicular bisector, we have that $OC = OA$. Finally, our previous triangle congruence establishes that $OF = OE$. Since these triangles are congruent, it follows that $EA = FC$.

We conclude that

$$BE + EA = BF + FC.$$

This is equivalent to the claim that $BA = BC$, which is to say that the hypotenuse of the triangle is equal to one of its legs. What went wrong?

SOLUTION: In drawing our original diagram, we simply assumed that O was inside the triangle. This assumption was false.



More precisely, O was the point of intersection of the angle bisector from B and the perpendicular bisector of side AC. It turns out that in any right triangle, this point lies outside the triangle, as we shall now prove.

In the figure to the left, we have circumscribed a circle around right triangle ABC. Standard results in geometry imply that AB must be a diameter, which implies that the midpoint of this segment must also be the center of the circle. Label this point K. If we construct segment KE perpendicular to AC, then an argument involving similar triangles shows that E is the midpoint of AC. That is, KE is the perpendicular bisector of AC. Extend KE to the point where it intersects the circle. Call this point O.

I claim that the angle bisector from B also passes through O. To see this, note first that O is the midpoint of the shorter arc joining A to C. This follows from the equality of the central angles $\angle OKC$ and $\angle OKA$. But the angle bisector from B must also pass through the midpoint of this arc, implying that it passes through point O as well. This completes the proof.

So, the "false proof" in the problem goes south in the very first step.