Problem of the Week
Solution Six

PROBLEM: Add the following list of number by calling out a running sum.

$$\begin{array}{r} 1000 \\ 10 \\ 1020 \\ 10 \\ 1030 \\ 10 \\ 1010 \\ 10 \end{array}$$

SOLUTION: The correct running sum goes like this: One thousand, one thousand ten, two thousand thirty, two thousand forty, three thousand seventy, three thousand eighty, four thousand ninety, four thousand one hundred.

That is, the correct sum is 4100. Most people call out 5000. This feels very natural, especially since you have been primed to think the number of thousands goes up one on every other line.

PROBLEM: I am going to prove that $4 = 0$. Consider the following system of equations:

$$\frac{x}{y} + \frac{y}{x} = 2$$
$$x - y = 4.$$

We can multiply the first equation by xy to obtain $x^2 + y^2 = 2xy$. This implies that $x^2 - 2xy + y^2 = 0$ and $(x - y)^2 = 0$. We conclude that $x - y = 0$, and finally that $x = y$.

If we now substitute into the second equation, we obtain $x - x = 4$. But $x - x = 0$. This is possible only if $4 = 0$, as claimed.

What went wrong?

SOLUTION: When substituting the results of the first equation into the second equation, we tacitly assumed that this system had a solution. Actually, we have effectively proved that there is no solution to this system. The graphs of these equations are parallel lines.

That seems surprising at first, since the first equation certainly does not look like a straight line. But it is, as our algebraic manipulations showed. It is the line $x = y$, with a “hole” at the point $(0, 0)$ (since our original fractions are not defined when either x or y is 0.