Problem of the Week
Solution Seven

PROBLEM: I am going to prove that $2 = 1$. Let x represent a positive number. We begin with the observation that $x^2 = x(x)$. We can expand the right-hand side to obtain

$$x^2 = x\underbrace{(1 + 1 + \cdots + 1)}_{x \text{ times}}.$$

Distributing on the right-hand side gives us

$$x^2 = \underbrace{x + x + \cdots + x}_{x \text{ times}}.$$

(For example, we can write $3^2 = 3(3) = 3(1 + 1 + 1) = 3 + 3 + 3$.) But now we can take derivatives on both sides to obtain

$$2x = \underbrace{(1 + 1 + \cdots + 1)}_{x \text{ times}}.$$

This implies that $2x = x$. Since x was assumed to be positive, we can divide by it to conclude that $2 = 1$

SOLUTION: We might first notice that the equation

$$x^2 = x\underbrace{(1 + 1 + \cdots + 1)}_{x \text{ times}}$$

is only correct when x is an integer. When x is not a whole number, it does not make sense to add 1 to itself x times. Consequently, it is just meaningless symbol manipulation to start taking derivatives of both sides.

That is a good observation, but there is another problem to consider. Let us suppose we could make sense of what it means to add x to itself x times even when x is not a whole number. There would still be the fact that our method for taking the derivative of the right-hand side was not correct. Each of the summands depends on x, and we took that into consideration when taking the derivative. The number of summands also depends on x, however, but we failed to take that into consideration in our derivative. That is another source of error in this argument.