Problem of the Week
Solution Eight

PROBLEM: I shall prove that $1 = 0$. To do that, I will use integration by parts to evaluate $\int \frac{1}{x}\,dx$. Make the following definitions:

$$u = \tfrac{1}{x} \qquad dv = dx$$

$$du = -\tfrac{1}{x^2} \qquad v = x$$

Recall that the formula for integration by parts is this:

$$\int u\,dv = uv - \int v\,du.$$

Applying this formula to the present case gives us:

$$\int \frac{1}{x}\,dx = \left(\frac{1}{x}\right)x - \int x\left(\frac{-1}{x^2}\right),$$

which simplifies to

$$1 + \int \frac{1}{x}\,dx.$$

Therefore, we have discovered that

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx.$$

Subtracting the integral from both sides leads to the conclusion that $0 = 1$, as claimed.

SOLUTION: Incredibly, everything was fine right up to the final step. The error came when we subtracted the integral from both sides. This is bad, because anti-derivatives are only defined up to an arbitrary constant. We know from elementary calculus that

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

So our final equation really just says

$$\ln|x| + C_1 = 1 + \ln|x| + C_2,$$

which is not contradictory at all.