Problem of the Week
Solution Seven
October 31, 2016

Jane has two clocks. She sets them accurately to the same time. After an hour has passed, she notices that the first clock gained one minute, while the second clock lost two minutes every hour. We shall assume that the clocks continue to run at a constant rate. The next morning she finds that the first clock reads 7:00 am, while the second clock reads 6:00 am. At what time did Jane set the two clocks?

SOLUTION: After each hour, the difference between the clocks grows by three minutes. Since the difference between them is now sixty minutes, we conclude that twenty hours have passed since the clocks were set. After twenty hours, the first clock has gained twenty minutes while the second clock has lost forty minutes. It follows that the correct time is now 6:40 am. Twenty hours before that is 10:40 am, and that is the answer.

At 6:00, the distance between the tips of the hands of a clock is 41. At 9:00 the distance between the tips of the hands of the same clock is 29. How far apart are the tips of the hands at 10:00?

SOLUTION: Let the lengths of the hands of the clock be denoted by x and y, with x denoting the minute hand and y denoting the hour hand. At 6:00 the distance between the tips is 41. That implies that $x + y = 41$. At 9:00 the hands of the clock form a right angle. Since the distance between the hands at this hour is given to be 29, we have that $x^2 + y^2 = 29^2$.

It is now a tedious, but doable, algebra exercise to solve for x and y. However, you might suspect that I would deliberately choose numbers that led to integer solutions. If you consult a table of Pythagorean triples, then you will find that $20^2 + 21^2 = 29^2$ gets the job done. So $x = 21$ and $y = 20$.

At 10:00, the angle between the hands is 60 degrees. We thus have sides of lengths 20 and 21, with an included angle of 60 degrees. If we denote by c the distance between the tips of the hands, then we use the law of cosines to compute

$$\begin{aligned} c^2 &= 20^2 + 21^2 - 2(20)(21)\cos 60 \\ &= 400 + 441 - 840\left(\frac{1}{2}\right) = 421. \end{aligned}$$

We conclude that $c = \sqrt{421}$.