
Problem of the Week

Number Three

September 18, 2017

More ten-themed goodness for you in this edition of POTW.

Number theory is, fundamentally, the study of the positive integers. Its practitioners prove theorems like this: Any prime that is one more than a multiple of four can be written as the sum of two squares, while any prime that is one less than a multiple of four cannot be so expressed. (For example, we have $13 = 4 + 9$ and $29 = 4 + 25$, but primes like 11 and 19 cannot be expressed as the sum of two squares.)

In their quest to understand the positive integers, mathematicians have defined many arithmetic functions. By this we mean a function whose domain is the positive integers.

One such function is denoted $\tau(n)$. It counts the number of divisors of n . Another is denoted by $\varphi(n)$. It counts the number of numbers smaller than, and relatively prime to, n . For example, the divisors of 15 are 1, 3, 5, and 15, which means that $\tau(15) = 4$. The numbers smaller than 15 that are relatively prime to 15 are 1, 2, 4, 7, 8, 11, 13, and 14, which means that $\varphi(15) = 8$.

What if we try this with 10? The divisors of 10 are 1, 2, 5, and 10. The numbers that are smaller than 10 and relatively prime to it are 1, 3, 7, and 9. That implies that

$$\tau(10) = \varphi(10) = 4.$$

Boom! The only other values of n for which $\tau(n) = \varphi(n)$ are 1, 3, 8, 18, 24, and 30. Very cool. Also cool is this week's problem, which is this:

The square of the number 999,999,999 is an 18-digit number. How many of these 18 digits are 9s?

What's that? You do not see any 10s in that problem? Well, perhaps they show up in the solution.

Of course, you could do this problem by laboriously computing

$$999,999,999 \times 999,999,999$$

but let me suggest there is a far more elegant approach.

When you think you have the problem figured out, follow the instructions below.

*Submissions are due to Jason Rosenhouse by 5:00 on **Friday, September 22**. Solutions, complete with a brief explanation, should be written on the back of an official POTW hand-out. Place your name, e-mail address, and the section numbers and professors of any math courses you are taking, in the **upper right corner** of the front of the page. One weekly winner will receive a five-dollar gift card from Starbucks. Solutions will be posted at the POTW website:*

[http://educ.jmu.edu/~rosenhjd/POTW/
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