Problem of the Week

Solution to Number Two

PROBLEM: Let ϕ be such that

$$\cos \phi = \sqrt{\sin^4 \theta + 4 \cos^2 \theta} - \sqrt{\cos^4 \theta + 4 \sin^2 \theta}.$$

Express ϕ in terms of θ.

SOLUTION: By making the substitutions

$$\sin^2 \theta = 1 - \cos^2 \theta \qquad \text{and} \qquad \cos^2 \theta = 1 - \sin^2 \theta$$

and multiplying out, we obtain

$$\begin{aligned}
\cos \phi &= \sqrt{\sin^4 \theta + 4(1 - \sin^2 \theta)} - \sqrt{\cos^4 \theta + 4(1 - \cos^2 \theta)} \\
&= \sqrt{\sin^4 \theta - 4\sin^2 \theta + 4} - \sqrt{\cos^4 \theta - 4\cos^2 \theta + 4} \\
&= \sqrt{(2 - \sin^2 \theta)^2} - \sqrt{(2 - \cos^2 \theta)^2} \\
&= (2 - \sin^2 \theta) - (2 - \cos^2 \theta) \\
&= \cos^2 \theta - \sin^2 \theta \\
&= \cos 2\theta.
\end{aligned}$$

It follows that we have
$$\phi = 2\theta.$$