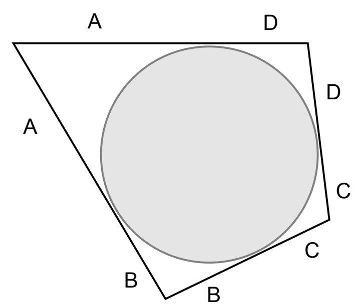
Problem of the Week
Solution Eight

In a quadrilateral which has an inscribed circle, the lengths of three consecutive sides are 4, 9 and 16 respectively. Find the length of the fourth side.

SOLUTION: The fourth side has length 11.

The main idea is to make repeated use of the theorem that says that given a circle and a point outside the circle, the two tangents drawn from the point to the circle have the same length. The diagram below, which depicts a quadrilateral with an inscribed circle, shows how we can make use of this theorem:



Let's assume that the unknown side is the one whose length is $A + B$. Then the two sides adjacent to it have lengths 4 and 16. The far side, which is not adjacent to it, must be the one with length 9. We can assume, without loss of generality, that the $B + C$ side is the one of length 4. Then our goal is to find $A + B$ given the system:

$$\begin{aligned} B + C &= 4 \\ C + D &= 9 \\ A + D &= 16. \end{aligned}$$

Adding the first and third equations gives $A + B + C + D = 20$. Using the second equation, we see that this simplifies to $A + B + 9 = 20$. Therefore, $A + B = 11$, as claimed.

It is interesting to note that the pairs of opposite sides of our quadrilateral have the same sum. (That is, $9 + 11 = 4 + 16$.) This is always true of a quadrilateral that has an inscribed circle, a result known as Pitot's theorem. It was proved by Henri Pitot in 1725. In fact, the argument we have given here is easily adapted to become a proof of Pitot's result. The converse, that a quadrilateral whose opposite sides have the same sum has an inscribed circle, is also true. This is harder to prove and was first shown by Jakob Steiner in 1846.