Problem of the Week
Solution One

PROBLEM: *Find four consecutive positive integers such that the sum of the cubes of three of them is equal to the cube of the fourth.*

SOLUTION: We find that

$$3^3 + 4^3 + 5^3 = 27 + 64 + 125 = 216 = 6^3$$

To show that this is the only possible solution, let us say that the four integers are given by $x - 1$, x, $x + 1$ and $x + 2$. Then we have:

$$(x - 1)^3 + x^3 + (x + 1)^3 = (x + 2)^3.$$

When everything is multiplied out and simplified we obtain the equation

$$x^3 - 3x^2 - 3x - 4 = 0$$

It is a basic fact about polynomials that the only possible integer solutions are found among those integers that divide the constant term. In this case, since we are looking specifically for positive solutions, the only possibilities are 1, 2, and 4. Trial and error quickly reveals that $x = 4$ is the only possible integer solution, which leads to the solution we gave above.