# Problem of the Week
# Solution Three

**PROBLEM:** *What is the sum of the digits in the number that is equal to the product:*

$$11 \times 101 \times 10{,}001 \times 100{,}000{,}001 \times 10{,}000{,}000{,}000{,}000{,}001$$

**SOLUTION:** The answer is $32$. Rewrite the product like this:

$$(10+1)(10^2+1)(10^4+1)(10^8+1)(10^{16}+1)$$

It is a basic fact that every integer can be expressed in a unique way as the sum of distinct powers of two. (This is known as the "dyadic expansion" of the number. Translating a number from base ten to base two is equivalent to finding its dyadic expansion.) It follows that when we expand this product, the result is

$$10^{31} + 10^{30} + 10^{29} + \cdots + 10^2 + 10 + 1$$

This is plainly a thirty-two digit number, all of whose digits are $1$. We conclude that the sum of the digits is $32$, as claimed.

If the business about dyadic expansions makes you uncomfortable, then here is another way of arriving at the answer. Suppose we multiply our product by $(10-1)$. Then everything collapses in a nice way. We have

$$(10-1)(10+1) = 10^2 - 1.$$

Then we have

$$(10^2-1)(10^2+1) = 10^4 - 1.$$

We continue in this way until we arrive at

$$(10^{16}-1)(10^{16}+1) = 10^{32} - 1.$$

This last number is plainly a sequence of $32$ nines. (Keep in mind that $10^{32}$ actually has $33$ digits.)

Since we started by multiplying by nine, we must now divide by nine to get back to our original product. The result of dividing $32$ nines by nine is a sequence of $32$ ones. So, again, the sum of the digits is $32$.