# Problem of the Week
# Solution Five

**PROBLEM:** *Find all positive integers $x$ for which*

$$(x-6)(x+14)$$

*is the square of an integer.*

**SOLUTION:** The only two solutions are $x = 6$ and $x = 22$. The first solution gives a value of $0$, while the second gives a value of $(16)(36) = 576$.

The solution $x = 6$ is obvious, since that makes the product equal to $0$. To look for other solutions, we start with the equation

$$(x-6)(x+14) = y^2.$$

We expand the left-hand side to get

$$x^2 + 8x - 84 = y^2.$$

By moving the $84$ to the other side and completing the square we obtain

$$x^2 + 8x + 16 = (x+4)^2 = y^2 + 100.$$

Now, if you notice that $100 = 10^2$, then our problem reduces to that of finding Pythagorean triples in which one of the legs is $10$. There is only one such triple, obtained from doubling the famous $5$, $12$, $13$ triple to obtain $10$, $24$, $26$. This implies that $x = 22$.

If you didn't happen to notice that, however, you could have proceeded as follows. We can perform the following algebra:

$$(x+4)^2 - y^2 = ((x+4)+y)((x+4)-y) = 100.$$

Thus, we now need to consider the various ways of factoring $100$. Trial and error will certainly work, since there aren't *that* many ways of factoring $100$, but we can actually narrow down the possibilities quickly. If we have that $x + 4 + y = a$ and $x + 4 - y = b$, then we immediately obtain that $2x + 8 = a + b$. Since $x$ is required to be a positive integer, we must have that $a + b$ is even. Since it is impossible to factor $100$ into the product of two odd integers, the only solutions will come from factoring $100$ into two even integers. Specifically, we must use $50$ and $2$, or $10$ and $10$.

If we set $x+4+y = 50$ and $x+4-y = 2$, then we immediately obtain $2x + 8 = 52$, and we find that $x = 22$. If we set $x + 4 + y = 10$ and $x + 4 - y = 10$, then we find that $x = 6$. All other ways of factoring $100$ will quickly lead to a non-integral value for $x$.