Problem of the Week

Solution Four

Find the (simplified) value of k for which the larger root of the equation $x^2 + 4x + k$ is

$$\left(\sqrt{2+\sqrt{3}}\right)\left(\sqrt{2+\sqrt{2+\sqrt{3}}}\right)\left(\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}\right)$$

Keep in mind that when using the square root sign, it is understood that it is the positive square root that is intended.

SOLUTION: The answer is $k = -5$.

To see what is going on, recall that $(a+b)(a-b) = a^2 - b^2$. Now look closely at the final two terms in the product. The terms underneath the big radical sign are conjugates. It follows that

$$\begin{aligned}\left(\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}\right) &= \sqrt{4-\left(2+\sqrt{2+\sqrt{3}}\right)} \\ &= \sqrt{2-\sqrt{2+\sqrt{3}}}.\end{aligned}$$

Do you see the point? We continue working our way to the left. We now compute:

$$\begin{aligned}\left(\sqrt{2+\sqrt{2+\sqrt{3}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{3}}}\right) &= \sqrt{4-\left(2+\sqrt{3}\right)} \\ &= \sqrt{2-\sqrt{3}}.\end{aligned}$$

The final multiplication gives us

$$\left(\sqrt{2+\sqrt{3}}\right)\left(\sqrt{2-\sqrt{3}}\right) = \sqrt{4-3} = \sqrt{1} = 1.$$

So after all of that, the larger root of the quadratic is 1. By plugging this into the quadratic we get $5 + k = 0$, from which the solution follows.