Problem of the Week
Solution Nine

Suppose that $f(x)$ and $g(x)$ are distinct linear functions. If we know that

$$f(f(x)) = g(g(x)) = 4x + 3,$$

then what is the product of $f(1)$ and $g(1)$?

SOLUTION: We will show that the product is -15.

Suppose that $f(x) = mx + b$. Then we have that

$$\begin{aligned} f(f(x)) &= m(mx + b) + b \\ &= m^2x + (bm + b) = 4x + 3. \end{aligned}$$

By equating coefficients on both sides, we get

$$\begin{aligned} m^2 &= 4 \\ bm + b &= 3 \end{aligned}$$

We see, therefore, that $m = \pm 2$. If we take $m = 2$, then we find from the second equation that $b = 1$. So, let us say that $f(x) = 2x + 1$.

If instead we use $m = -2$, then we find that $b = -3$. Thus, we have that $g(x) = -2x - 3$.

Now it is easy to compute that $f(1) = 3$ and $g(1) = -5$, so the answer is $f(1)g(1) = -15$, as claimed.