Chapter 1

What is Discrete Mathematics?

To understand what discrete mathematics is and why people regard it as important, we must begin by understanding what mathematics is intended to do. To gain such understanding we must first give some thought to the nature of scientific investigation.

1.1 Science

In ancient times the workings of the world were largely mysterious. Despite this, it was perfectly obvious that nature was not entirely capricious or arbitrary. The Sun rises each morning and sets each night. The moon is sometimes full and sometimes a crescent, but these phases occur in an orderly succession. Seasons change in predictable ways. An object dropped from a moderate height falls to the ground in a straight line, accelerating as it does so. Seeds that receive copious water and sunlight grow into plants while those deprived of water and sunlight do not.

At times, however, nature did seem capricious and whimsical. Rain is essential to any agricultural economy, but the Ancients had not the faintest idea how to predict its coming. Certainly the Sun rose and set, but occasionally things would grow very dark for no apparent reason (a phenomenon now known as an eclipse). Though it was known that tiny seeds grew into plants, the process by which they did so may as well have been magic.

Since caprice and whimsy are attributes of intelligent agents, and since no Earth-bound intelligences seemed up to the task of creating or maintaining nature, the Ancients tended to invoke the action of gods to explain what was mysterious. Sadly, such explanations suffer from an obvious practical problem. It is entirely possible that the rains come and eclipses occur because it amuses the gods to provide rain and eclipses, but nothing that was formerly confusing becomes comprehensible by such an explanation. Appealing to the whimsy of the gods is no help when you are trying to determine when to plant your crops. No disease was ever cured by attributing its cause to the action of demons. No useful bit of technology has ever been invented by viewing nature as whimsical and capricious.

Over the years people realized that a more useful sort of explanation was possible, and it was out of this realization that science was born. The goal of scientific exploration was to render nature predictable and controllable. Given that goal, what sorts of things would we need to do to have any chance of success? Certain things will occur to you immediately. We will need extensive observations of what actually happens in nature, and we will need to record the data we collect. Measuring devices of various sorts are necessary (indeed, the practical difficulties involved in constructing good measuring devices stymied scientific progress for quite some time). Since it is sometimes difficult to make good observations "in the wild", it is necessary to conduct experiments in a controlled setting.

But there is one further ingredient that may not occur to you so readily. Let us suppose our goal is to describe the motion of a baseball thrown from our hand. Everyday experience tells us the ball's path depends on many factors. Some are obvious: the angle and velocity at which the ball leaves your hand, the force of gravity, and the effect of air resistance, for example. Others are less so, like the rotation of the Earth or the small gravitational pull of the Moon. If all of these factors must be integrated into our explanation, there is little hope of producing something useful. It is necessary instead to focus only on the variables we consider most essential to predicting the motion of the ball. Consequently, the missing ingredient is ***abstraction***.

1.2 Abstraction

Abstraction (from Latin words meaning "to draw away from") is the process whereby we remove from consideration many of the variables that affect real-world observations. We focus instead on a small number of remaining variables in the hope that it is precisely those variables that are most important. As a practical matter, perfect precision is not necessary in predicting the motion of the ball. We are therefore justified in ignoring those variables having little effect on the ball's path.

For example, we know that the gravitational pull of the Moon is so small that ignoring it is not likely to affect things too badly. Similarly, the error introduced by ignoring the Earth's rotation will be negligible. Since a baseball is reasonably small and dense the effect of air resistance will be small as well. In the end we are left only with the angle and velocity at which the ball leaves our hand, and the force of gravity. So small a number of variables can be handled without too much difficulty, and there is some hope of producing a successful theory from this starting point.

Students confronting mathematics for the first time often cite this reliance on abstraction as the main stumbling block to learning the subject. The point is well-taken. Many of the abstractions mathematicians study seem so divorced from any real-world situation that they can be difficult to understand. In another sense, however, abstraction is precisely what makes mathematics possible at all. The alternative to abstraction is a consideration of every real-world factor that has any effect, no matter how small, on the question we are studying. As we have seen, that is not practical. In other words, abstractions are simple; it's the real world that's complicated!

Let us return now to our thrown baseball. We are assuming that it leaves my hand at a known angle and a known velocity, and that after leaving my hand the only force acting on it is that of gravity. The goal is to determine how far the ball will travel and the highest point it will reach along the way. How shall we reach our goal?

One of the first people to study this question in any detail was Isaac Newton in the seventeenth century. It was his realization that predicting the path of the baseball depended not on a deep understanding of baseballs, but rather on a deep understanding of the curves describing the motion of thrown objects. We can imagine a graph with time along the x-axis and position along the y-axis. From everyday experience we know that the ball will initially rise, eventually reach some maximum height, and then begin to fall. By applying certain basic ideas from physics and mathematics, Newton realized that the curve that emerges from this trajectory is a parabola (from Greek words meaning "put alongside" or "thrown right up to", the idea being that the parabola resides at the boundary between the ellipse and the hyperbola in the theory of conic sections. The Greek can also be translated as "to compare", and a story that compares a simple situation to a more complex one is called a parable for that reason). Thus, Newton set about discerning the properties of parabolas.

The collection of methods Newton developed for solving this and related problems are nowadays referred to as calculus.

1.3 Pure vs. Applied Mathematics

The parabola studied by Newton is one example of a continuous function; "function" because it describes a precise relationship between two variables (position and time), "continuous" because the curve can be drawn without removing your pencil from the paper. The continuity of the parabola reflects the fact that our baseball, in moving from point A to point B, must pass through every point in between A and B. Thus, if at one moment the ball is five feet above the ground and at some later moment it is seven feet above the ground, you can be sure there was some in-between moment when the ball was six feet above the ground.

There are many everyday quantities that change in a continuous way. If you are driving at forty miles per hour at noon and ten minutes later you are driving at sixty miles per hour, then at some point during those ten minutes your were driving at fifty miles per hour. If you suddenly slam on your brakes, thereby changing your speed to zero, then you can be sure that for any number (and by number we do not necessarily mean an integer) between sixty and zero, there was a moment in time when you were traveling at exactly that speed.

Temperature has this property as well. In heating water from fifty to sixty degrees Celsius, you can not avoid a moment at which the water was at fifty-seven degrees, or any other temperature between fifty and sixty.

If you start with an empty tub and begin filling it with water, you can not go from no water to ten gallons of water in one shot. You must pass through every intermediate value, including fractional values, between zero and ten gallons. Gas pressure has this property as well. No doubt you can think of other examples.

If all of these real-world quantities can be described by continuous functions, it stands to reason that the methods Newton developed in solving the baseball problem might find wider applicability. In fact, we might even infer there is something to be gained from studying continuous functions for their own sake, with the idea that whatever wisdom we gain will inevitably be applicable to *something*. This sort of study is sometimes referred to as pure mathematics. It is to be contrasted with applied mathematics, in which abstractions are constructed and studied in the hopes of solving a particular real-world problem. The line between pure and applied mathematics is sometimes blurry, but it is real nonetheless.

In emphasizing the potential future usefulness of pure mathematics, I do not mean to imply that practical utility is the sole reason for doing mathematics. Indeed, for most mathematicians the utility of the subject is a marvelous side benefit, not the primary motivation. The joy of doing mathematics is found in the beauty and elegance of the subject. The satisfaction of encountering the mysterious and, using nothing more formidable than the power of your own intellect, making it comprehensible, is something that must be experienced to be properly understood.

In this book our focus will be primarily on pure mathematics. Though we will discuss many applications along the way, our attitude will be that certain abstractions have proven to be so useful in so many situations that they are worth studying for their own sake.

1.4 Discrete Mathematics

We are now ready to answer the question asked in the title of this chapter.

At this point it should be clear that there are many real-world quantities that change in a continuous manner. But it is equally clear that there are certain other quantities that do not change in this way. You can purchase twelve eggs or six eggs or eight eggs, but you can not buy seven and a half eggs. If their growth is left unchecked, the number of bacteria in a Petri dish will double in every generation. Thus, in consecutive generations you will have one, two, four and eight bacteria, without ever having five or six bacteria. In logic, statements are either true or false with nothing in between. The electrons of a particular atom reside at certain specific energy levels, and they can jump from one level to another without ever occupying a level intermediate between the two.

These sorts of quantities are said to be discrete (from Latin words meaning “kept apart”), and the branch of mathematics devoted to their study is discrete mathematics. Since discrete structures lack the property of continuity, we can not hope to study them via the techniques of calculus. Instead we need new techniques and new abstractions. Developing such techniques is the main goal of this book.

The study of discrete mathematics enjoyed a renaissance in the mid-twentieth century. This resulted from the sudden importance of computers in day-to-day life. You see, most of the things physicists study are continuous in the sense described earlier. It is for that reason that they use the techniques of calculus so frequently. Computers, however, brought with them a wealth of new problems to be solved.

Here is one example: In programming a computer to carry out some task you begin by devising an algorithm for the computer to follow. By this we mean a step-by-step description of what you will have the machine do as it executes your program. It is possible, however, that the algorithm you devise requires too much time or too much memory to be practical. To determine whether this is the case, an assessment must be made regarding the complexity of the algorithm, and the first step in doing this is to count the number of individual computations the computer is required to carry out. Counting is a discrete process.

This nicely illustrates the idea that not only do mathematical discoveries routinely affect the prevailing culture, but also the culture affects the mathematics that is studied.

1.5 Doing Mathematics

How, exactly, are we to study these abstractions?

If a group of scientists decides they want to study the physical properties of baseballs, they are not hindered by a lack of agreement as to what a baseball is. We all know a baseball when we see one. A scientist need only point to a ball and say "That is the object we are studying."

But the whole point of an abstraction is that it is not a real world object. It is an imaginary construct, one that exists only in the mind of the mathematician proposing it. How, then, are two different mathematicians to be certain they are studying the same abstraction?

Certainly we must begin with a precise definition of the abstraction in question. This is not always so easy. If I want to define the word "blue", I can do so by pointing to a large number of blue objects in the hope that whoever is listening to me will get the point. Similarly, if I want to define the word "chair" I can do so by pointing to a large number of chairs.

But mathematical abstractions are not like that. There are no real world objects for me to point at when stating my definition. Consequently, a mathematical definition must be written with a level of precision that is unfamiliar from everyday life. Like the abstractions themselves, this level of precision is a formidable stumbling block for many students of mathematics. For example, earlier I said that a continuous function is one whose graph can be drawn without lifting one's pencil from the paper. This captures the idea behind continuity, but as a mathematical definition it is not adequate. There are many functions whose graphs are not easily drawn. Is there no hope of determining whether they are continuous? And if you are trying to prove that an abstractly defined function is actually continuous, it is hard to imagine ending your proof with the statement, "We see, therefore, that the graph of this function can be drawn without lifting your pencil from the paper." The actual definition of continuity, perhaps familiar to you from your calculus classes, is so complex that mathematics instructors must work very hard to establish its connection with the more intuitive meaning of the word. But this level of precision is necessary if we are to reason properly about abstract objects.

Incidentally, we should not get too carried away with the extent to which abstractions have no real world existence. While it is true that, technically, they don't exist, it is equally true that all of the abstractions we will be concerned with in this book were motivated by common, everyday objects and ideas. These motivations will be a powerful guide for keeping our bearings in the mathematical wilderness. For example, a perfect circle is an abstraction found nowhere on Earth, but there are plenty of objects that come close enough.

Having defined our objects, the next question is how to determine what is true about them. When dealing with physical objects I can experiment with them to my heart's content. Thus, if I drop my baseballs from a known height and find that they always reach the ground in the same amount of time, I can conclude that I have learned something true about the world. There may be a nagging doubt that the next experiment will be the one that causes my theory to collapse, and this is why all scientific theories are held tentatively. But the fact remains that this sort of reasoning has worked very well in the past. Sadly, experiments of this sort are precisely what we can not do with mathematical abstractions. Since the abstractions themselves have no physical existence, there is no way to conduct experiments upon them.

(Here, again, we should not get too carried away. Though we can not perform experiments in the same sense that scientists perform experiments, we can do the next best thing. Specifically, we can work out specific examples in the hopes of discerning a general pattern. This sort of experimentation is a standard part of mathematical reasoning, and we will be doing a great deal of it throughout this book.)

If I can't determine the truth about my abstractions by experimentation then I will have to try a different approach. The information contained in my definitions is all I have to work with when drawing conclusions about my abstractions. What is needed, therefore, are some basic principles that will tell me how to draw correct inferences from the limited information I have. The branch of mathematics (or philosophy, depending on your perspective) that provides such principles is known as logic, and that will be the subject of the next chapter. Having decided on some rules of inference, we can then address the question of what constitutes a proof in mathematics. That will be the subjects of chapters three and four.