Chapter 5

Counting

Now that we have armed ourselves with the requisite logical machinery, we can go on to consider some of those useful abstractions we mentioned in chapter one. Since the first thing most people think of upon hearing the word "mathematics" are the counting numbers (positive integers), we will devote this chapter to the subject of counting.

5.1 Sets

As a first attempt at imposing order on nature's chaos, let us note that certain real-world objects can be grouped together in a natural way. Apples are easily distinguished from oranges, and dogs are quite different from cats. As I look around my office I notice that there are some red objects, some green objects and some blue objects. The books on my shelf are distinct from the books on the shelf of my neighbor's office, and I can make this distinction by referring to "My books" vs. "His books."

Mathematicians capture the idea of grouping similar objects together to distinguish them from other objects in the concept of a ***set***. By a set we mean any collection of objects. We can talk about the set of all books in my office or the set of all things that are red or any other grouping of objects you care to imagine. In a mathematical context we often talk about sets of numbers. Thus, I can discuss the set of positive integers, or the set of even integers, or perhaps the set of rational numbers.

I could also talk about smaller sets. In that case, rather than write down a one-sentence description of the set I have in mind, as I did in the previous paragraph, I can simply list all the elements of the set. For example, if I am thinking of the set containing all of the positive integers that are no larger than five, I could write down my set like this:

$$\{1, 2, 3, 4, 5\}.$$

When writing down a set, it is customary to place the items in the set within braces, and to separate those items with commas. It is also customary to refer to the items in a set as the ***elements*** of that set. Thus, the integers between one and five inclusive are the elements of the set above.

So we have now imposed a small degree of order upon the numerous objects we confront in day-to-day life. But I am sure that you have noticed by now that some objects answer to more than one description. Perhaps I have the set of all apples *over here* and the set of all things that are red *over there*. These two sets are not entirely separate. Apples come in a variety of colors and one of those colors is red. Such apples reside comfortably in both sets.

This is a common phenomenon. I can talk about the set of all even numbers and the set of all perfect squares, but the fact remains that some numbers (like 4, 16 and 36) answer to both descriptions (they are even on the one hand, and perfect squares on the other). Or I can look at a more mundane example. Let A be the set $\{1, 2, 3, 4, 5\}$ and let B be the set $\{3, 5, 7, 9, 11\}$. Then a quick inspection reveals that the numbers 3 and 5 are found in both.

From this we conclude that it is possible for two sets to overlap. This overlap is referred to as the ***intersection*** of the two sets. Since "intersection" is a long, clumsy word, it is customary to abbreviate it with the symbol "$\cap$". In the paragraph above, we could say

$$A \cap B = \{3, 5\}.$$

The expression on the left hand side here is read "A intersect B". We could also say that the intersection of the set of all apples with the set of all red things is the set of all red apples. Please notice that the intersection of two sets is, itself, a set. Sets that have no elements in common (for example, the set of all even numbers and the set of all odd numbers), are said to be ***disjoint***.

This raises an interesting dilemma. As just noted, the intersection of two sets is itself a set. But what happens if the sets in question have no elements in common. For example, if E is the set of even numbers and O is the set of odd numbers, then $E \cap O$ contains no elements (since there are no numbers that are simultaneously even and odd). It follows that we need a name for the set that has no elements. Since such a set can plausibly be said to be empty, we refer to the set that contains no elements as the ***empty set***. This set is denoted by the symbol "$\emptyset$." Thus, we would say

$$E \cap O = \emptyset.$$

We have now identified, for any two given sets A and B, a smaller set containing precisely those elements common to both A and B. But we could also identify a larger set containing everything that appears either in A or B or both. This larger set is referred to as the ***union*** of A and B, and we write $A \cup B$. For example, using the notation from the previous paragraph, we can say that $E \cup O$ is the set of all integers, since every integer is either even or odd. Or, to consider a more mundane example, if we have

$$A = \{1, 2, 3, 4, 5\} \text{ and } B = \{3, 5, 7, 9, 11\},$$

then we can say that

$$A \cap B = \{3, 5\} \text{ and } A \cup B = \{1, 2, 3, 4, 5, 7, 9, 11\}.$$

Notice that the union of two sets is another set, and that we do not list the elements 3 and 5 twice, even though they appear in both sets.

Forming the union of two sets has some obvious real-world counterparts. In an elementary school it is sometimes convenient to talk about the set of third graders, the set of fourth graders and the set of fifth graders. At other times I may want to take the union of these sets and talk simply about the set of all students at the school. Or I can view each particular kind of apple (Red Delicious, McIntosh, Granny Smith, and so on) as its own set, and then take their union to obtain the set of all apples. No doubt you can think of other examples.

You have probably noticed that it is possible to have two sets such that one of them lies entirely inside the other. For example, the set of all even integers sits inside the set of all integers. The set of all integers, in turn, sits entirely inside the set of all rational numbers (since every integer is a rational number, but there are rational numbers that are not integers). Non-mathematical examples are possible as well. The set of all Golden Delicious apples sits inside the set of all apples, for example. The set of all math majors taking my course lies entirely inside the set of all math majors at my university.

Since the idea of one set residing entirely inside another is dreadfully imprecise, mathematicians talk instead about one set being a ***subset*** of another. We say A is a subset of B if all of the elements of A are also elements of B. For example, we could say the set of even numbers is a subset of the set of all integers. We could write this more succinctly as $A \subseteq B$, the symbol "$\subseteq$" being an abbreviation for "is a subset of".

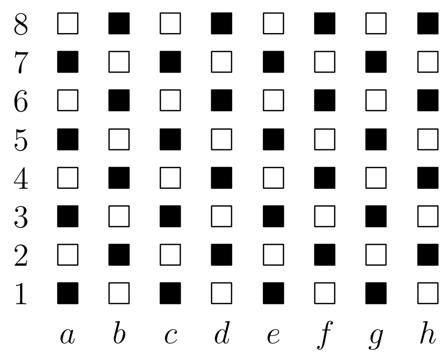
There are many other sorts of things you can do with sets, but perhaps it is time to stop and ponder why we are bothering at all. Why have we developed this elaborate formalism for something as simple as grouping similar objects together? Well, in solving the various counting problems that will form the basis for the remainder of this chapter, we do not want to overspecialize by considering concrete examples of sets. By this I mean we do not want to consider sets of books, or sets of people, or sets of numbers, for example, because doing so puts unnecessary restrictions on our reasoning. By considering sets in the abstract we can be sure that the principles we develop will be applicable to any kinds of sets we wind up encountering.

This principle will become clearer as we proceed with the chapter.

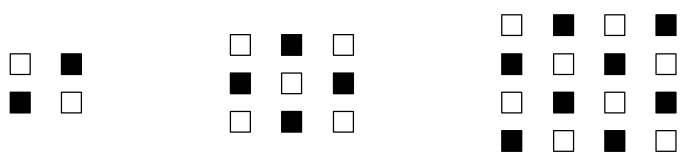
5.2 Two Classics

Counting is the art of putting the elements of a given set into one-one correspondence with the elements of a set whose size is known. Sometimes this is easily done. If you have before you a basket of apples and wish to determine how many there are, you can simply remove the apples one at a time, ticking off the numbers one, two, three as you do so. Unless we are confronted with a truly enormous basket, this is a very practical procedure. Saying there are x apples in the basket is equivalent to saying the apples in the basket can be placed into one-one correspondence with the set $\{1, 2, 3, \ldots, x\}$.

But what if our interest lies not in physical objects, but rather in the elements of some abstractly defined set? For example, suppose our goal is to count the number of squares, of any size, in an 8×8 checkerboard:



We can see at a glance there are sixty-four small squares. But what about the squares of other sizes, like the 2×2, 3×3, and 4×4 squares below:



It would seem that tackling the problem all at once is rather too complicated. There are so many squares to be counted that, lacking some orderly procedure for listing them all, we will quickly lose track of which we have counted and which we have not counted. What is needed is some way of breaking the problem down into more manageable chunks.

What if instead of counting all of the numbers at once, we instead restrict our attention to those squares of a particular size? We have already seen there are sixty-four small, 1×1 squares. How many 2×2 squares are there?

To answer that we might notice that every 2×2 square in the diagram has exactly one lower left corner, and any 1×1 square can be the lower left corner of only one 2×2 square. It follows that we can determine the number of 2×2 squares by counting instead the number of 1×1 squares that can serve as a lower left corner. Inspecting the diagram above reveals that any square other than those in row eight or column h can serve as such a corner. It follows there are 49 squares of dimension 2×2 in the diagram.

Imitating this logic for the case of 3×3 squares leads to the observation that any square except for those in rows seven or eight, or those in columns g and h, can serve as the lower left corner of a 3×3 square. Consequently, there are 36 such squares. Perhaps the pattern is now clear? The number of squares of size $n \times n$, where n is some integer between one and eight inclusive, is given by $(9-n)^2$. For example, the number of 3×3 squares is $(9-3)^2 = 36$. It follows that the total number of squares in the diagram is given by

$$\sum_{k=1}^{8}(9-k)^2 = \sum_{k=1}^{8} k^2 = 204.$$

Notice the manner in which we rewrote the sum after the first equal sign above. As k goes from one to eight, the quantity $(9-k)$ counts backward from its first term, which is eight, to its last term, which is one. From this we see that the quantities k and $(9-k)$ cycle through the same list of numbers.

The nice thing about this method is that it generalizes easily to checkerboards of any arbitrary size. If we have a checkerboard of size $d \times d$, then the total number of squares it contains is given by

$$\sum_{k=1}^{d}(d+1-k)^2 = \sum_{k=1}^{d} k^2 = \frac{d(d+1)(2d+1)}{6},$$

the last equality having been proved in the previous chapter.

This is a classic sort of counting problem. By judicious application of our intellect we were able to take a chaotic jumble of squares and impose some order upon them.

For our second example, consider a single-elimination tennis tournament with $1024 = 2^{10}$ players. By single-elimination we mean that as soon as a player loses a match he is eliminated from the tournament. The tournament continues until only one player remains. The question is: how many matches are played in the tournament?

Brute force seems workable in this case. We notice that in the first round the 1024 players will engage in 512 matches. In the next round the remaining 512 players will engage in 256 matches. The next round will feature 128 matches, then 64 in the round after that, and so on. Then it is a simple computation to show that the total number of matches is

$$512 + 256 + 128 + \ldots + 4 + 2 + 1 = 1023.$$

We could even generalize this process to the case where instead of having 2^{10} players, we have 2^n players instead. We would then observe that the number of matches in the k-th round of such a tournament would feature 2^{n-k} matches. Then the total number of matches would be

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1.$$

This last formula could be proved by induction.

Clever as all this is, it simply does not work when the number of players in the tournament is not a power of two. In this case certain rounds will feature an odd number of players, and we must assume the tournament organizers have worked out some system of byes to handle these rounds. But we can still ask how many matches will be played before a winner emerges.

Initially the situation appears hopeless. Since we know nothing about the number of players we have to begin with, there is no hope of using any sort of computational approach.

But perhaps all is not yet lost. What if instead of counting the number of matches, we counted instead some other set that had a size equal to the number of matches? There is such a set, which you will realize for yourself as soon as you consider that every match produces one winner and one loser. Since there is only one winner at the end of the tournament, there must be $x - 1$ losers. And since every match produces only one loser, we find there must have been $x - 1$ matches played. This is precisely what we found in the special case where the number of players was a power of two.

5.3 Basic Counting Principles

There are many problems in mathematics that can be solved merely by implementing some appropriate algorithm. No doubt you could add together two numbers when called upon to do so, and you would only have to work slightly harder to add two fractions. These problems are simple because there is a very clear procedure for solving them

Not so with counting problems. Indeed, not so with any mathematical problem worth spending effort to solve. This does not mean we must begin from scratch with each new problem to come down the pike, however. When a plumber arrives to fix your sink, he comes prepared with the standard tools of his trade. He may not know specifically which tools he will need to solve your problem, but he has a good sense of what tools have been most helpful in the past. Furthermore, his experience enables him to determine quickly which tool is likely to be useful in his new situation.

So it is with mathematics. When approaching a new problem a mathematician does not know ahead of time what approach will work. But he carries with him a collection of techniques and ideas that have often been useful in the past. It is part of the training of any mathematician to solve many sorts of problems, and the experience gained thereby allows him to discern very quickly what tools to use in new situations.

Since we can not write down a general procedure for solving counting problems, we will do the next best thing. We will come up with basic principles to guide our investigations into these sorts of problems. We have already encountered two:

1. One way to count the elements of a large set is to partition it into smaller sets whose elements are easier to count.
2. You can count the number of elements in a set by placing it into one-one correspondence with a set whose size is known.

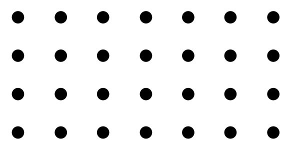
In item one above, the term "partition" indicates that every element in the larger set is found in exactly one of the smaller sets. It was this fact that justified our concluding step, wherein we summed the number of elements of the smaller sets to find the size of the large set. Any elements residing in more than one of the smaller sets would have otherwise been overcounted in our sum, you see.

If we so choose we can view item one from the reverse perspective. Rather than partition a large set into smaller, disjoint sets, what if we start with several disjoint sets and seek the number of elements in their union? It will come as no surprise that the logic does not change, leading us to our third principle:

3. If $S_1, S_2, \ldots, S_n$ are disjoint sets, then
$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{i=1}^{n} |S_i|.$$

This is a concise way of saying that the number of elements in the union of disjoint sets is found by adding the number of elements in each set taken individually.

Let us conclude this section with one further example. Suppose your goal is to count the number of dots in this rectangular array:



You could achieve your goal by counting every dot, one by one, moving your finger across each row and ticking off the numbers as you do so. This method is unappealing, since aside from being terrifically dull it is also prone to error; your finger is no doubt thicker than the individual dots, making it difficult to keep your place. A far more practical method begins with the observation that there are seven dots along the bottom row and four dots in the first column. It ends by noting that $7 \times 4 = 28$.

Now imagine that along the bottom row we placed the letters a through i, while along the first column we placed the numbers 1 through 4. In that case we could identify each dot in the array with a pair (x, y) where x represents one of the letters a through i, and y represents one of the numbers between 1 and 4. Therefore, we conclude there are 28 such pairs.

This fact can be expressed more generally. If A and B are finite sets, then we can define their ***Cartesian product*** to be the set whose elements are the pairs (a, b), where a is an arbitrary element of A and b is likewise arbitrarily chosen from B. The Cartesian product of A and B is denoted $A \times B$, read "A cross B". We can imagine placing the elements of A along the bottom of a rectangular array and the elements of B along the left most column of the same array. Having done that, every element of $A \times B$ will be associated with exactly one dot in the array. This leads us to our fourth basic counting principle:

4. Let A and B be sets. Assume $n(A) = x$ and $n(B) = y$. Then

$$n(A \times B) = xy$$

5.4 The Pigeonhole Principle

Since we are presently discussing basic principles, we should also mention that if you have more pigeons than pigeonholes then some pigeonhole contains more than one pigeon. This self-evident truth is known as the pigeonhole principle. It is also known more grandiosely as the Dirichlet principle, after the eighteenth century French mathematician who first coined it. We can express it more formally by saying that if n objects must be distributed among fewer than n spots, then some spot receives more than one object.

If more precision is required we might say that if n and k are positive integers such that $nk + 1$ objects must be distributed among k spots, then some spot receives are least $n + 1$ objects. For example, given eleven objects and five spots, some spot receives at least three objects. In this case, $n = 2$ and $k = 5$.

An amusing illustration of the usefulness of this principle is offered by the following problem: let us suppose we are given six points in the plane, no three of which are collinear. Suppose that all pairs of points are joined by line segments, and that each of these line segments is colored either red or green. Is it necessarily the case that some three of the six points form the vertices of a triangle all of whose sides are the same color?

To solve this, let $P_1, \ldots, P_6$ be the six points. Then the pigeonhole principle implies that three of the five segments having P_1 as an endpoint must have the same color (we have five segments but only two colors, you see). Let us assume that color is red, and let us further assume that P_2, P_3 and P_4 are the second endpoints of these segments. Consider the triangle whose vertices are these three points. If this triangle has all green sides, then we have found a triangle of the type we seek. So let us suppose that one of the triangle's sides is red, say, the one joining P_2 to P_3. In that case, the triangle joining the points P_1, P_2 and P_3 is the triangle we seek. Either way there is a triangle all of whose sides are the same color. Consequently, the answer to our question is yes.

5.5 Permutations and Combinations

If a certain ice-cream parlor offers thirty-one flavors, each of which can come in one of four different types of cone, then there are $31 \times 4 = 124$ different items you can order from them. This is easily seen by noting that for each of the thirty-one flavors there are four serving vessels from which to choose. Alternatively, we might define F to be the set of available flavors and V to be the available serving vessels. Then each item you might order is given by an ordered pair (f, v), where f is a flavor and v is a serving vessel. It follows that the number of items available at the ice cream parlor is equal to the number of elements in $F \times V$. Since F contains thirty-one elements and V contains four elements, we again arrive at 124 different items.

What if I now tell you there are ten different toppings to choose from as well? No doubt you would reason that for each of my 124 flavor plus cone combinations, I have ten choices of topping. This leads to $1,240$ possibilities in all.

Though we are using nothing more sophisticated than the definition of multiplication, a surprising number of counting problems can be solved by this sort of reasoning. The idea is that if you want to count the number of ways of carrying out some multi-step process, it is helpful to count the number of ways of carrying-out each step individually. In the example above, we sought the number of ways of ordering an ice cream cone from our hypothetical parlor. Ordering such a cone was a three step process: first specify a flavor, then specify a type of cone, and finish by specifying a topping.

Let us express this principle in a more general setting. Suppose we are carrying out some process that has k steps. Further suppose there are p_1 different ways of carrying out the first step, p_2 ways of carrying out the second step, and so on through p_k possibilities for the k-th step. Then there are $p_1 p_2 \dots p_k$ ways of carrying out the entire process. This can be proved by induction, but it can also be seen by generalizing the reasoning that went into showing that the number of elements in the Cartesian product of two sets is given by the product of the number of elements in each set individually.

As an example, let us suppose we desire to count the number of different ways in which the integers from 1 through 9 can be ordered. We imagine that we have nine slots before us, each to be filled with one of our integers. That means we can place any of the nine integers in our first slot. Once we have filled that slot, there will be eight integers and eight slots remaining. It follows that the second slot can be filled in any of eight different ways. Continuing in this way, we find there will be seven ways to fill the third slot, six ways to fill the fourth slot, and so on until we have only a single possibility for the final slot. We conclude that the number we seek is:

$$9 \times 8 \times \dots \times 2 \times 1 = 9!$$

The abbreviation 9! is read "Nine factorial" and indicates that we are to multiply all the integers between one and nine inclusive. In general, the number of ways of ordering k distinct objects is $k!$ (read "k factorial" and indicating that we are to multiply all of the numbers between one and k together).

What if we have n objects to choose from, but we only need to fill r slots, where $r < n$? Our reasoning would not change at all, save for the fact that now we will run out of slots before we run out of objects. The number we seek is:

$$n \times (n-1) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}.$$

As another example of this line of reasoning, suppose that the license plates in a certain state consist of three letters followed by three digits. How many distinct license plates can the state make? To solve this, we observe there are six slots to fill. Each of the three letter slots can be filled in twenty-six ways, while each of the number slots can be filled in ten ways. We conclude that the answer is:

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = (26^3)(10^3) = 17,576,000.$$

Now let us add a new wrinkle. In this example, two license plates differing only in the order in which the letters and numbers are arranged are nonetheless viewed as different license plates. In other words, the plate ABC 123 is considered to be different from CBA 231, and both of these are different from ACB 132.

Orderings of this sort, in which the order of the chosen symbols matters, are known as ***permutations***, from Latin words referring to changes or movement. They are to be contrasted with ***combinations***, where our interest lies solely in the chosen objects themselves, and not in the order in which we choose them.

To make this distinction clear, suppose we are to count the number of ways we might choose three numbers out of the ten digits from 0 to 9. We might reason that since we are choosing three objects from a total of ten possibilities, the formula we derived above tells us the answer is

$$\frac{10!}{(10-3)!} = 10 \times 9 \times 8 = 720.$$

By arguing in this way, we are assuming that the order in which we choose the three numbers ought to affect our count. Thus, for example, we have been treating the sequences $(1, 3, 4)$, $(4, 3, 1)$ and $(3, 4, 1)$ as being different.

But what happens if we ignore the order in which we choose the numbers? In that case the three sequences above would be treated as identical. It follows that our old formula overcounts the number we seek. What to do?

Well, to what extent does our old formula overcount the number we seek? We find that once we have chosen any three numbers, they can be ordered in six different ways. This follows from our general observation that k distinct objects can be ordered in $k!$ different ways. If the three numbers we had chosen were 1, 3 and 4, then in addition to the three orderings given above we also have

$$(1,4,3) \qquad (3,1,4) \qquad (4,1,3),$$

for a total of six. It follows that our formula counts each sequence six times. Therefore, the number we calculate from our formula will be precisely six times greater than the number we seek. We can compensate for this by dividing our answer by six, giving us $\frac{720}{6} = 120$.

More generally, let us suppose we are choosing r objects out of n possibilities, where we do not care about the order in which we choose the objects. We will denote this number by the symbol $\binom{n}{r}$ (read "n choose r"). There are $\dfrac{n!}{(n-r)!}$ ways of choosing the r objects if the order is considered to be important. Having selected the r objects, we find they can be ordered in $r!$ different ways. It follows that

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}.$$

For example, suppose we have fifteen people. Our task is to choose five of those people to be part of a committee. In how many ways can we make the choice? Since the order in which our five people are chosen makes no difference to the composition of the resulting committee, we find there are

$$\binom{15}{5} = \frac{15!}{10!\,5!} = \frac{15 \times 14 \times 13 \times 12 \times 11}{5 \times 4 \times 3 \times 2} = 273$$

possible committees.

As another example, in a game of bridge you are dealt thirteen cards. The order in which the cards are dealt to you makes no difference. Since there are 52 cards in a deck, we find there are

$$\binom{52}{13} = \frac{52!}{39!\,13!}$$

possible bridge hands, which is a large number indeed.

5.6 Poker Hands

Let us apply some of our new-found wisdom to the vexing problem of discerning the proper hierarchy of poker hands.

In poker, each player receives five cards, referred to as his hand. These hands are ranked hierarchically so that given any pair of hands, one of them can be said to be stronger than the other.

Certain types of hands are of particular interest, and among these are the flush, in which all five cards are of the same suit; the straight, in which the numerical values of the five cards comprise five consecutive numbers (counting the jack as eleven, the queen as twelve, and the king as thirteen, with aces acting either as low card or high card); and the full house, in which three of the cards are of one denomination, while the remaining two cards are of some other denomination. The strength of a poker hand is inversely proportional to the probability of obtaining it; the more improbable the hand the higher ranked it is.

Many novice players find it difficult to recall the proper ordering of the flush, straight and full house, so we settle that question now. Doing so will require us to combine several different counting techniques.

Let us begin with the number of ways of obtaining a flush. The difficulty here lies in finding an orderly way of counting all the possible flushes. This is reminiscent of the checkerboard problem we considered earlier, and that problem was solved by partitioning the set to be counted into disjoint sets whose size was more easily determined.

We can do the same thing here. Since counting all of the flushes is too complicated, let us instead count all of the flushes of a particular suit, say diamonds. We then need to choose five specific diamonds out of the thirteen available. Since the order in which the cards are listed does not change the hand, we find there are $\binom{13}{5}$ ways of choosing the five cards. And since there are four possible suits, the answer is:

$$4\binom{13}{5} = \frac{(4)(13!)}{8!5!} = 5,148.$$

What about the straight? Here we might follow the example of the tennis tournament problem considered earlier. Rather than count the number of straights directly, we notice that the every straight has exactly one low card. Once we have chosen the value of the low card, the values of the remaining four cards are determined. We observe that any card from ace up to ten can serve as the low card of a straight, while cards higher than ten can not. Thus, the low card in the straight can have any of ten possible values.

Let us suppose that we have chosen three to be the low card. Having made that choice, we see that the remaining cards must be a four, a five, a six and a seven. Therefore, we have five slots to fill in our hand and four choices for each slot (since there are four cards in the deck of any given value). It follows that there are 4^5 straights having three as the smallest card. And since we had ten choices for what the lowest card would be, we find that the number of possible straights is

$$(10)(4^5) = 10,240.$$

There are nearly twice as many straights as there are flushes.

What about the full house? The key here is to break the problem down into more manageable steps. In a full house all five cards come from just two values. Since there are 13 possible values, I might begin by observing that there are $\binom{13}{2}$ ways of choosing two of them. However, given any pair of values, there are two sorts of full houses that can be made out of them. For example, if I wanted to have a full house with kings and sixes, I could have either three kings and two sixes, or two kings and three sixes.

The next observation is that, having chosen my values, I must now choose three representatives of one of the values and two representatives of the other value. There will be $\binom{4}{3}$ and $\binom{4}{2}$ ways of making those choices respectively. Putting everything together gives us:

$$2\binom{13}{2}\binom{4}{3}\binom{4}{2} = (2)(78)(4)(6) = 3,744$$

possible full houses.

That was sufficiently complicated that it may help to look at it another way. We have constructed our full houses via a three-step process. In the first step, we chose the two values to be represented in the full house, taking into consideration the fact that for any two values there were two different full houses to be built upon them. There are $2\binom{13}{2}$ ways of completing step one. In step two we chose three representatives of one of the values. There are $\binom{4}{3}$ ways of completing step two. Step three involved choosing two representatives of the second value. There are $\binom{4}{2}$ ways of completing step three. Since we know the number of ways of carrying out each step in the process we can simply multiply them together to find the answer.

As a result, we see that a flush is stronger than a straight, and a full house is stronger than a flush.

The numbers $\binom{n}{k}$ are fascinating in their own right, and they will figure prominently in chapter six.

5.7 Problem Solving Skills

5.7.1 Permutation Variations

Your ability to solve counting problems of the sort considered in this chapter will depend as much on your persistence and ingenuity as it will on your ability to master specific techniques. With that in mind, let us consider some variations on the sorts of problems considered above.

How many ways are there of arranging the letters in the word GOAT? This is identical to the problem of counting the number of ways of arranging the digits one through four. We have already seen there are $4! = 24$ such orderings. For completeness, let us list the twenty-four ways of ordering the four letters in GOAT:

GOAT, GOTA, GTOA, GTAO, GAOT, GATO
OGTA, OGAT, OTGA, OTAG, OATG, OAGT
TAGO, TAOG, TOGA, TOAG, TGAO, TGOA
ATGO, ATOG, AOGT, AOTG, AGTO, AGOT

Now let us try a harder problem. How many ways are there of arranging the letters in the word MATHEMATICS? At first this may seem to be identical to the previous problem. If it were, we would observe that MATHEMATICS is an eleven letter word and conclude there are 11! different orderings. This would be the correct answer if the word contained eleven different letters

But since the letters M, A and T all appear twice in the word "mathematics", this would be an overcount. For example, let us distinguish the occurrences of the letter M by using a capital letter for its first occurrence and a lower case letter for its second. In offering 11! as the correct answer we are effectively treating the orderings MATHEmATICS and mATHEMATICS as being different. In reality, however, the two occurrences of the letter M are identical. So we will incorrectly count this particular ordering twice.

How should we handle this? We might begin by asking how badly we overcounted in suggesting 11! as the correct answer. To answer this, we continue with the reasoning from the previous paragraph. Given any particular ordering of the eleven letters, we can obtain an identical ordering simply be reversing the capital and lower-case M's. A proper count would treat these orderings as identical, whereas our 11! treats them as different. We can compensate for this by dividing the 11! by two.

From there the rest follows easily. Since the positions of the two A's can be switched without producing a new ordering, we conclude that we must again divide by two to compensate. Finally, we divide by two again to take into consideration our overcounting of the T's. When we are done, we find the answer is $\frac{11!}{8}$.

Actually, there was another way to approach this problem. This time we begin by noticing that we have eleven slots to fill. We must choose two of those slots to contain the M's, but we do not care about the order in which these two slots are chosen. Consequently, there are $\binom{11}{2}$ ways to choose the locations of the two M's. Having made that choice, there are now nine slots remaining to be filled. Two of those must be given to the A's, and again we do not care about the order in which those two slots are chosen. It follows that there are $\binom{9}{2}$ ways of making that choice. We repeat the process one more time for the T's, and find that there are $\binom{7}{2}$ ways of making that choice.

At this point we have filled six of the eleven slots. The remaining slots must be filled with our remaining five letters. These letters are all distinct (we have one each of H, E, I, C and S). Therefore, we have 5! ways of ordering them. Putting everything together, we have taken the problem of ordering the letters in MATHEMATICS and have broken it down into a four-step process. It follows that the total number of orderings is then given by:

$$\binom{11}{2}\binom{9}{2}\binom{7}{2}5!$$

By doing a little algebra we obtain:

$$\left(\frac{11!}{2!\ 9!}\right)\left(\frac{9!}{2!\ 7!}\right)\left(\frac{7!}{2!\ 5!}\right)5! = \frac{11!}{2!\ 2!\ 2!} = \frac{11!}{8},$$

which is is exactly the answer we obtained before.

5.7.2 Binomial Coefficients

Numbers of the form $\binom{n}{k}$ are known as ***binomial coefficients***, for reasons that will be made clear in the next chapter. A surprising number of problems can be solved simply by remembering that $\binom{n}{k}$ counts the number of ways of choosing k objects out of n possibilities, where we do not care about the order in which we choose the objects. For example, suppose we want to compute the sum

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}.$$

Viewed as an abstract computational problem it is hard to see where to begin. But what if we approach this as a counting problem? In that case we can reinterpret our sum as follows: Suppose I begin with n objects. Then $\binom{n}{k}$ counts the number of ways of selecting k elements from the n possibilities. This is equivalent to counting the number of subsets of size k that can be formed from a set with n elements. Furthermore, the size of any subset of a set of size n will have between zero and n elements in it. By reasoning in this way, we see that the sum above is counting the total number of subsets possessed by a set with n elements.

In the previous chapter we showed that a set of size n has 2^n subsets. It follows that our sum is equal to 2^n. For example, suppose that $n = 3$. Then we observe that

$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$

and

$$1 + 3 + 3 + 1 = 8 = 2^3.$$

Since the argument we provided for solving this problem depended on recognizing that our sum was counting a familiar object (the number of subsets of a given set in this case), we refer to it as a ***counting argument***.

As another example, suppose we wish to prove that if n, k and r are positive integers with $n > k > r$ then

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$

Certainly this can be done algebraically. By this I mean that we could replace the four binomial coefficients with the appropriate fractions and, by a judicious application of algebra, show that the two expressions really are equal.

That would establish the correctness of the equation, but it is not a very enlightening proof. By this I mean that it provides little insight into why the equation is true. Mathematical objects may be abstract, but they are supposed to be telling us something about real-world objects. Perhaps we can show this equation is true by relating it to some real-world situation. Why did it ever occur to anyone to investigate this equation?

Examine the left-hand side. It is telling us first to choose k objects out of n possibilities. Having done that, we then choose r objects out of the k we just chose. We might think of this as first forming a committee with k members out of n people to choose from. We then form a subcommittee of r members out of the k people available.

Now examine the right-hand side. Does it also count the number of ways of forming a subcommittee of size r out of a group of k people, who were themselves chosen from a group of n people? The first term on the right is $\binom{n}{r}$, and it represents the number of ways of choosing r people out of n possibilities. It is as if we are first choosing the r people for the subcommittee. The second term tells us to choose $k - r$ people out of the remaining $n - r$ possibilities. After we have done that, k people will have been chosen in total.

Summing up, on the left-hand side we first form a committee of size k and then, from those k, form a subcommittee of size r. On the right-hand side we begin by choosing the subcommittee of size r, and then choose $k - r$ additional people to enlarge our subcommittee into a committee of size k. It follows that the equation must really be true.

A far more enlightening way of looking at the problem, don't you think?

5.8 Problems

1. In how many ways can the letters in the word POWER be ordered?

2. In how many ways can the letters in the word BOOKKEEPER be ordered?

3. In how many ways can the letters in the word MISSISSIPPI be ordered?

4. At a certain college the food plan allows students to choose up to

three pieces of fruit each day. The fruits available are apples, bananas, peaches, pears, plums, nectarines, and pineapples. If a student takes his maximum allotment of fruit each day, for how many days can he make a different selection? How would your answer change if instead of choosing exactly three pieces of fruit each day, the student chooses no more than three each day (so that the number of pieces chosen each day is some number between zero and three)?

5. In how many ways can four distinct letters be chosen from the alphabet, if we require that the letters be chosen in alphabetical order?

6. Suppose you are given a standard deck of 52 cards. Further suppose that you draw two cards at random from the deck. In how many ways can you draw a card that is either a king or a spade? In how many ways can you draw a three or a seven? How many cards would you have to choose to be certain that some three of them were of the same suit?

7. Suppose that a bookshelf is to be used to display nine new books. Further suppose that there are eight computer science books and seven mathematics books from which to choose. If we decide to display four mathematics books and five computer science books, and are required to keep the books in each subject together on the shelf, how many different displays are possible?

8. Determine the number of ways of obtaining the following poker hands: One pair, Two pairs, Three of a kind, Four of a kind and Straight flush.

9. How many strings of six distinct lowercase letters from the English alphabet contain the letter a?

10. How many zeros appear at the end of (100!)?

11. Suppose you are taking a true/false test with twenty questions. In how many ways can you answer the questions if every question is answered? How would your answer change if we allowed the possibility of leaving some of the questions blank (so that there are three possible answers to each question: True, False and Blank).

12. What is the least number of three digit area codes needed to guarantee that forty million phones have distinct, ten-digit phone numbers?

13. In how many ways can we seat eight people in a line if there are two people who refuse to sit next to each other?

14. Suppose you choose five points at random in the interior of a square whose sides have length two. Prove that there are two points whose distance from each other is smaller than $\sqrt{2}$.

15. How many positive integers would you need to be certain that at least six of them leave the same remainder when divided by seventeen?

16. In how many ways can the letters A through E be placed around a circle? By this we mean that orderings like (A, B, C, D, E) and (B, C, D, E, A) should be considered identical. In general, two circular arrangements are the same if one can be obtained from the other by a clockwise rotation. Can you generalize your result to include the possibility of n objects arranged in a circle, instead of just five?

17. Let us define the sum of a set of integers to be the sum of the elements in that set. Let S be a set of positive integers, none of them greater than 15. Suppose that no two disjoint subsets of S have the same sum. Prove that S can not contain more than 5 elements.

18. When you write out the numbers from one to one thousand, how many times do you use the number five?

19. Suppose each square of a 3×7 grid is colored either black or white. One example of such a grid is the following:

$$\begin{array}{cccccccc}
3 & \square & \square & \blacksquare & \square & \blacksquare & \blacksquare & \blacksquare \\
2 & \blacksquare & \square & \square & \square & \blacksquare & \square & \square \\
1 & \square & \blacksquare & \blacksquare & \blacksquare & \square & \blacksquare & \square \\
 & a & b & c & d & e & f & g
\end{array}$$

Prove that in any such grid there is always a rectangle with four distinct squares as corners such that those corners are all the same color. In the example above, the rectangle whose corners are $b2$, $b3$, $d2$ and $d3$ is one such rectangle.

20. Let n be a positive, even integer. Provide a counting argument to prove the following equation:

$$\sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} = \sum_{k=1}^{\frac{n}{2}} \binom{n}{2k-1}.$$