Chapter 8

Functions and Relations

In chapter five we identified sets as a first attempt to impose order on nature's chaos. The idea was to group together objects answering to some common description, thereby distinguishing them from objects not answering to that description. In this chapter we will push these ideas a bit farther.

The basic intuition running through this chapter is that sometimes elements of a set share a certain relationship to each other, or to elements in a different set. For example, we know that the real numbers have an order relation defined on them. Given two distinct real numbers, we can always declare that one of them must be smaller than the other. Taking note of this relation is useful in a number of mathematical contexts. Another example is the divisor relation on the positive integers. We might say that integers a and b are related if a divides b. This sort of language is useful when we want to single out a property of interest shared by certain elements of a large set.

We want to examine this notion in more detail. We begin with the notion of partitioning a set.

8.1 Partitions

It is sometimes convenient to break up a large set of objects into smaller, more manageable subsets. Universities often divide their students into freshman, sophomores, juniors and seniors. When a list of objects is placed in alphabetical order, they are effectively divided into groups depending on the letter of the alphabet with which they begin. The land mass comprising the United States is helpfully divided into fifty states, and these states are further divided into smaller counties.

These are all examples of **partitions** of sets. In each case our large set is divided into smaller subsets. Every element in the large set appears in exactly one of the subsets. Since that is the crucial defining feature of a partition, we pause to examine it more closely.

In dividing students into freshman, sophomores, juniors and seniors, it is understood that every student belongs to at least one of those categories. That is, if you are a student at the university, then you must be either a freshman, sophomore, junior or senior. Furthermore, it is not possible to belong simultaneously to two different categories. You can not be a freshman at the same time you are a sophomore, for example. Therefore, these four subsets partition the set of students at a university. Another example along the same lines is the division of younger students into grades.

Likewise for the states comprising the United States. The states do not overlap. There is no piece of land that is simultaneously in two different states. On the other hand, every piece of land that is part of the U.S. lies in a state. So the individual states partition the land mass of the U.S. Likewise for the counties comprising the individual states. Every plot of land in the state resides in some county, and no piece of land resides in two counties simultaneously.

The idea of partitioning a set seems rather natural, and the reader is invited to devise further examples. Since our primary interest in this class lies with sets of numbers, let us consider a few mathematical examples.

1. The set of positive integers can be partitioned into the even numbers and the odd numbers. Every positive integer is either even or odd, and no positive integer is simultaneously both even and odd.

2. The real numbers can be partitioned into the rational numbers and the irrational numbers. Since the set of irrational numbers is defined to be the set of real numbers that are not rational, it is clear both that every real number is either rational or irrational, and that no number appears in both sets.

3. Denote by $\overline{k}$ the set of positive integers that leave a remainder of k when divided by six. For example, we have $\overline{1} = \{1, 7, 13, 19, 25, 31, \cdots\}$ and $\overline{4} = \{4, 10, 16, 22, 28, \cdots\}$. Then the sets $\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}$ form a partition of the positive integers.

4. Sticking with the positive integers, the set of even integers and the set of perfect squares do not form a partition. First, there are numbers such as 17 that belong to neither set. It is not even and it is not a perfect square. Second, there are some numbers, like 36, that appear in both sets.

5. Let $\mathbb{Q}_k$ denote the set of rational numbers with a denominator of k when written in lowest terms (where we adopt the convention that negative rational numbers are to be written as the ratio of a negative numerator to a positive denominator). Then the sets $\mathbb{Q}_k$ partition $\mathbb{Q}$.

6. Let $S = \{1, 8, 17, 23, 45, 79, 89, 93\}$. Then we can partition S as follows:

$$\{1, 8, 17\}, \quad \{23, 45\}, \quad \{79, 89, 93\}.$$

Alternatively, we could partition S like this:

$$\{1\}, \quad \{8, 23, 89\}, \quad \{17, 45\}, \quad \{79, 93\},$$

or like this

$$\{1, 8\}, \quad \{17\}, \quad \{23\}, \quad \{45\}, \quad \{79\}, \quad \{89, 93\}.$$

A number of points arise from these examples. The first is that there is more than one way to partition a set. Consequently, it is important that you never refer to "the partition" of S. Rather, it is always "a partition" of S (unless, of course, you are working in a situation where a specific partition can be taken for granted).

Next, in talking about partitions it is important to use words properly. The individual subsets comprising a partition should be referred to as the **parts** of the partition. Thus, when I partition the positive integers into evens and odds, that is one partition with two parts. Return to example six above. When I say S can be partitioned into

$$\{1, 8, 17\}, \quad \{23, 45\}, \quad \{79, 89, 93\},$$

I have defined a single partition. That partition comes in three parts. In example six we defined three different partitions of S. The first had three parts, the second had four parts and the third had six parts.

I mention this because students, in learning about partitions for the first time, have a tendency to refer to the subsets themselves as "partitions." That is, the partition given above is sometimes carelessly referred to as "three partitions of S." Resist this tendency! In mathematics, even more so than in most other disciplines, it is essential to express yourself accurately.

Finally, look again at example five. Since k can be any one of infinitely many positive integers, we see that our partition of $\mathbb{Q}$ in that example has infinitely many parts. That is acceptable. There is no rule that says we are allowed only finitely many parts. So long as every element in the set belongs to exactly one of the subsets, you have a perfectly good partition.

8.2 Relations

With the exception of number six, all of the partitions in the examples above were obtained by singling out some property of interest among the elements of the set under consideration. This property gave us a criterion for determining whether two elements belonged to the same part of the partition. Thus, in example one we placed two integers in the same part if both were even or both were odd. In part two we placed two real numbers in the same part if both were rational or both were irrational. In part three the question was whether two given integers left the same remainder upon division by six. And so on. The point is that we stipulated some property of interest, and then used that property to induce a partition on the set.

This process is described succinctly by the statement, "We are defining a **relation** on the set S." That is, if $a, b \in S$, then we say a is related to b if they satisfy whatever condition we choose to lay out. In example one we would say that two integers are related if both are even or both are odd. In example five we said the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are related if they have the same denominator when written in lowest terms. This sort of language applies very generally. You are limited only by your imagination in devising criteria for relatedness among elements in a set.

Two points are crucial. First, the statement, "a is related to b," is entirely distinct from the statement "b is related to a." There is no guarantee that either one implies the other. For example, suppose we define the following relation on the set of positive integers: a is related to b if $a < b$. Then we have that 5 is related to 7 (since $5 < 7$), but 7 is not related to 5 (since it is not the case that $7 < 5$). In fact, this relation has the property that the statement "a is related to b," implies the statement "b is not related to a."

The second point is that a and b are not required to be distinct. That is, it is reasonable to ask if an element in a set is related to itself. It may be, or it may not be, depending on the relation.

Any partition of a set leads naturally to a particular relation. Given a partition of a set, we can say that two elements x and y are related if they lie in the same part of the partition. Conversely, the examples above show that sometimes a relation on a set can lead to a partition. The relation on the integers that declares x and y to be related if both are even or both are odd is an example. Let us examine this process more closely.

How, exactly, do we go from a particular relation to a partition of a set? After all, talking about partitions only makes sense in the context of an entire set. A family of subsets is a partition only if every element in the set lies in exactly one of the subsets. A relation, by contrast, is something we discuss in the context of pairs of elements. We say elements x and y are related if they satisfy the particular test we have established. How do we pass from the local phenomenon of a relation, to the global phenomenon of a partition?

The relation in which integers x and y are related if they have the same parity partitions the set of integers into the evens and the odds. Every even number is related to every other even number, and no even number is related to any odd number.

What about the relation that declares two students to be related if both are in the same year? This relation partitions the set of students into the sets of freshman, sophomores, juniors and seniors. All of the freshman are related to each other, and no freshman is related to anyone in the other sets. Likewise for the sophomores, juniors and seniors.

This suggests a procedure for going from a particular relation to a partition of a set. Let us suppose that S is a set with a relation defined on it. We choose arbitrarily an element $x \in S$. We then consider the set of all of the elements of S that are related to x. Let us denote that set as $[x]$. Now we find an element $y \in S$ with $y \notin [x]$, assuming there is such an element, and construct the set $[y]$. We continue in this way until we exhaust the elements in S.

This procedure may or may not lead to a partition of S, depending on the nature of the relation. Let us apply this procedure to some of our familiar examples:

1. Consider the set of positive integers with the relation x is related to

y if both are even or both are odd. Let us start with the integer 1. Since 1 is odd, the set of things related to it are precisely the set of odd integers. Therefore, 1 is related to 1, because both of those numbers are indeed odd. Now all of the odd numbers are accounted for, but there are integers not related to 1. The integer 2 is one such. The set of things related to 2 contains all even numbers, including 2 itself. And now we are finished. Every integer is related either to 1 or to 2, and no number is related to both of them. So we have obtained a partition.

2. Let S be a set of university students. We consider the relation "A is related to B if both are in the same year." Following the procedure described above, choose an arbitrary student X. Since X must be in some year of college, we can assume without loss of generality that X is a freshman. The subset of S containing all of the things related to X will be precisely the set of all freshmen in S. Notice that X is an element of this set. After all, X and X are trivially in the same year. So the set of things related to X will contain all of the freshmen in S. Anyone remaining must be either a sophomore, junior or senior. We choose another person Y, and assume without loss of generality that Y is a sophomore. Then the set of things related to Y will contain all sophomores in S, including Y. Continuing in this way we see that we do, indeed, obtain a partition. Applying the procedure described above leads naturally to the partition of a set of university students into freshmen, sophomores, juniors and seniors.

3. Return now to the set of positive integers. Suppose we say x is related to y if both leave the same remainder when divided by six. Following the procedure described above, we start by choosing an arbitrary integer, say 14. We note that 14 leaves a remainder of 2 when divided by six. Therefore, the set of integers related to 14 will consist of all integers that leave a remainder of 2 when divided by six. This includes 14 itself. There are still many integers not included in this set, such as 27. This number leaves a remainder of 3 when divided by 6. It is related to all positive integers that leave a remainder of three. This will lead to a partition as well.

4. Sticking with the positive integers, define the relation x is related to y if $x > y$. If we try to apply our procedure for generating partitions we quickly encounter a problem. The number 1 is not related to anything.

According to our criterion, if 1 were related to some positive integer x, then we would have to have $1 > x$. But there are no positive integers less than 1.

This is enough to show that we will not obtain a partition in this case. Since 1 is not related to anything, it will not be included anywhere. But let us push the analysis a bit further. The set of positive integers related to 10 is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. According to our procedure, we would use this as one of the parts of our partition. The next step involves selecting an integer not included in this set, such as 12. The set of things related to 12 is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. This would also be a part in our partition. This is a problem, however, since the parts of a partition have to be disjoint.

Our procedure breaks down on two fronts. First, there is an element in the set that is not related to anything, and therefore will be left out of the partition. Second, the sets we obtain in attempting to produce a partition are not disjoint.

5. Now consider the following relation on the positive integers: x is related to y if x divides y. We would have, for example, that 3 is related to 15, and 5 is related to 40, but 37 is not related to 56. This time we have that everything is related to *something*, since any integer divides itself. Sadly, the good news ends there. Note that the only number related to 1 is 1 itself, so $\{1\}$ would be one part of our attempted partition. But we also have that the numbers that divide two are 1 and 2. That means that $\{1, 2\}$ would also be in the partition. Since these sets are not disjoint, the procedure will not work in this case.

6. We consider one final example. Suppose that positive integers x and y are related if either one divides the other. As in the previous example, we have that every integer is related to itself. Now, if x is any integer, what does $[x]$ look like? Certainly x is related to all multiples of itself. It will also be related to each of its divisors. For example, we would have $[12] = \{1, 2, 3, 4, 6, 12, 24, 36, 48, \cdots\}$. This does not lead to a partition, however, since we also have $6 \in [18]$. Since $[12] \neq [18]$, we have found two different sets with a nontrivial intersection.

It is clear that some relations lead naturally to partitions of sets while others do not, and it would be nice if we had some method for distinguishing between them. Deriving such a method will be the main task of the next section.

8.3 Equivalence Relations

Our task is to devise a list of necessary and sufficient conditions for a relation to lead naturally to a partition. We will use our examples from the last section as a guide. So, let S be a set with a relation defined on it. If $x, y \in S$ are related, then we will write $x \sim y$.

Our first condition is inspired by example four:

1. If $x \in S$ then there is an element $y \in S$ such that $x \sim y$.

This is clearly necessary. If there are elements in S unrelated to anything else in S, then those elements will not appear in any of the sets we hope will form parts of our partition.

Example four showed something else that can go wrong. If $x \sim y$ but $y \not\sim x$, then it is impossible to have $[x] = [y]$. Since such an eventuality would prevent us from obtaining a partition, we must address this in our list of conditions. So we add a second condition:

2. If $x, y \in S$ and $x \sim y$, then $y \sim x$.

The relation in example six satisfies both of our first two conditions but does not lead to a partition. The problem lay in the possibility of having two integers related to the same thing, that nonetheless were not related to each other. For example, we had $18 \sim 6$ and $6 \sim 24$, but we did not have $18 \sim 24$. Consequently, we have $18, 24 \in [6]$, but $18 \not\sim 24$. We add one further condition to deal with this possibility:

3. If $x, y, z \in S$ and $x \sim y$ and $y \sim z$, then $x \sim z$.

It seems that we have now exhausted the sort of pathologies we noted in the previous section. Indeed, we can now prove the following theorem:

Theorem 1. *Let S be a set with a relation defined on it. Assume the relation satisfies conditions one through three above. Let $x, y \in S$. Define*

$$[x] = \{z \in S \mid x \sim z\}.$$

Then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

Proof. Condition one guarantees that for all $x \in S$, we have that x is related to some other element of S. Therefore, $[x] \neq \emptyset$. Now let $x, y \in S$. To prove the theorem we will assume $[x] \cap [y] \neq \emptyset$, and show that this implies that $[x] = [y]$.

Suppose $[x] \cap [y] \neq \emptyset$. Then there exists some element $z \in ([x] \cap [y])$. It follows from the definition of $[x]$ that $z \sim x$ and $z \sim y$. By condition two, the fact that $z \sim x$ implies $x \sim z$. And by condition three, we have that $x \sim z$ and $z \sim y$ implies that $x \sim y$.

I claim that this implies $[x] = [y]$. Suppose not. Then we can assume without loss of generality that there exists an element $w \in S$ with the property that $w \in [x]$ but $w \notin [y]$. But then we would have $w \sim x$ and $x \sim y$, but $w \not\sim y$, which is a contradiction of property three. Therefore, if $[x] \cap [y] \neq \emptyset$, then we must have $[x] = [y]$. □

So we have achieved our goal. We have a list of necessary and sufficient conditions for determining when a particular relation on a set leads to a partition. Given a relation satisfying the three properties above, we can generate a partition via the procedure described in the previous section.

We now pause to note an interesting fact. Any relation satisfying the three properties above must also satisfy the following property:

4. If $x \in S$, then $x \sim x$.

In the exercises you will be asked to prove this fact. Since condition four is a bit simpler than condition one, we will use it as a replacement for condition one on our list. We did not do this initially simply because condition four is less general than condition one, and it is good to prove theorems with as much generality as possible.

We close this section with some definitions.

Definition 1. *Let S be a set with a relation defined on it.*

1. *If for all $x \in S$ we have that $x \sim x$, then we say the relation is* ***reflexive****.*

2. *If for all $x, y \in S$ we have that $x \sim y$ implies $y \sim x$, then we say the relation is* ***symmetric****.*

3. *If for all $x, y, z \in S$ we have that $x \sim y$ and $y \sim z$ implies $x \sim z$, then we say the relation is* ***transitive****.*

A relation that is reflexive, symmetric and transitive is said to be an ***equivalence relation****. If $x \in S$ and $[x]$ is the set of all elements of S that are equivalent to x, then we say that $[x]$ is the equivalence class of x.*

Thus, the equivalence classes of an equivalence relation partition the set. Note, incidentally, that two different elements in the set can generate the same equivalence class. In fact if we have $x \sim y$ then $[x] = [y]$.

8.4 More General Kinds of Relations

Thus far our discussion of relations has focused on the idea that certain elements in a set might have some connection to each other that is not shared by the other elements of the set. Since recognizing such connections is an important part of doing mathematics, it seemed reasonable to devote a few pages to this subject.

Sometimes, however, it is convenient to discuss the idea of relatedness without having a specific relation in mind. That is, we want a notion of relatedness that does not depend on having to stipulate a particular property of interest. We want to discuss the idea of a relation on a set in the abstract.

This is easily done. We can simply define a relation on a set S to be a subset of $S \times S$. Recall that $S \times S$ is the set of all ordered pairs, both of whose entries are drawn from S. Thus, a relation on a set is simply a set of ordered pairs with entries drawn from the set.

The relations we have considered in previous sections can be viewed in this way. For example, the relation on the integers given by "x is related to y if both are even or both are odd," is simply the collection of ordered pairs

$$(2,4),(2,6),(1,3),(17,19),(24,26),(15,51),\ldots$$

and all the other ordered pairs of integers in which both components have the same parity. We interpret each pair as saying that the first coordinate is related to the second coordinate.

Viewed this way, it is the list of ordered pairs themselves that comprise the relation. Where the pairs came from is no longer our concern. For example, let $S = \{1,2,3,4,5,6,7,8\}$. We can define a relation R on S as follows:

$$R = \{(1,2),(7,4),(2,3),(3,2),(6,6)\}.$$

That set of ordered pairs is a relation on S. It was not constructed by identifying a particular property of interest, but it is a perfectly good relation nonetheless.

A reflexive relation on S would be a set of ordered pairs that contains (x, x), for all $x \in S$. The relation is symmetric if for every pair $(x, y) \in R$, we also have that $(y, x) \in R$. And it is transitive if any time the pairs (x, y) and (y, z) are both in R, the pair (x, z) is in R as well.

There is also no reason to restrict our attention to relations defined on a single set. Sometimes you are confronted with two different sets and wish to identify some relationship between them. For example, perhaps you are planning a seating arrangement for a wedding. You have the set of guests at the wedding on the one hand and the set of tables in the reception area on the other. A relation between these sets might consist of pairing up each guest with the table at which they will be seated.

This leads us to our most general notion of relation:

Definition 2. *Let S_1 and S_2 be sets. A* ***relation*** *from S_1 to S_2 is a subset of $S_1 \times S_2$.*

Of course, if $S_1 \neq S_2$ then it is no longer meaningful to discuss the notions of reflexivity, symmetry and transitivity. On the other hand, relations between different sets might have other properties of interest, and we turn to that subject in the next section.

8.5 Functions

For our purposes a function is a special kind of relation. Let S_1 and S_2 be sets and let R be a relation between them. Then R is a subset of $S_1 \times S_2$; it is a set of ordered pairs whose first component comes from S_1 and whose second component comes from S_2. We say that R is a **function** if every element of S_1 appears exactly once as the first component of an ordered pair in R. In this case we say S_1 is the **domain** of the function and S_2 is the **codomain** of the function.

A synonym for function is **mapping**. Given a function f with the property $f(x) = y$, it will be convenient to say "f maps x to y."

As an example, consider the relation from the last section in which each of the guests at a wedding reception is paired with the number of the table at which they will be seated. This relation is a function. Since every guest will be sitting somewhere, we see that each guest will show up as the first coordinate of at least one pair. On the other hand, since no guest will be sitting at two tables simultaneously, we can be sure that no guest will show up as the first coordinate of more than one pair.

No doubt you are already familiar with functions from previous math classes. In calculus, for example, you routinely study functions like $f(x) = x^2$. This is a function from the real numbers to the real numbers. The function defines a set of ordered pairs, with each component of the pair being a real number. Among the pairs defined by this function are

$$(1, 1), (2, 4), (3, 9), (\pi, \pi^2), (-2, 4)$$

and so on. Since every real number has exactly one square, we see that $f(x)$ satisfies our definition.

Notice something important in the two examples we have considered. The definition of a function requires that every element of S_1 appear exactly once as the first component of a pair. On the other hand, no restriction is placed on which elements of S_2 appear as the second component of such a pair. In the example of the wedding guests, we can assume that no guest will be placed at a table by himself. Consequently, each of the table numbers is likely to appear multiple times as the second component of a pair. Likewise for the example of $f(x) = x^2$. The same real number can appear as the second component of more than one pair.

You should also notice that we do not require every element of the codomain to appear as the second component of a pair in the function. In the example of $f(x) = x^2$, the negative real numbers do not appear in that role. The subset of the codomain containing precisely those elements that appear as the second component of some pair, is called the **range** of the function.

If the intuition behind the idea of a relation was that elements of a given set might have connections to one another that differentiate them from the other elements of the set, then the intuition behind the idea of a function is that sometimes the value of one quantity can be expressed in terms of another. If you are driving at a constant velocity, then the total distance you have traveled can be expressed in terms of the total time you have been traveling. This relationship can be expressed by the formula $d = rt$, where d is the distance you have traveled, r is your velocity, and t is the time during which you were driving. You might also say that distance is a function of time.

When a flask full of gas is heated, the pressure the gas applies against the inside walls of the flask can be expressed as a function of the temperature. In physics, the gravitational force between two objects of fixed, given mass can be expressed as a function of the distance between them. The volume of a cube is a function of its side length. You might amuse yourself by coming up with further examples.

8.6 One-To-One, Onto and Invertible Functions

In studying relations we identified certain properties a relation might have that turn out to be especially interesting. We now do the same for functions.

We begin by defining some notation. If f is a function with domain S_1 and codomain S_2, then we will write $f : S_1 \to S_2$.

Let $x, y \in S_1$. If the statement $f(x) = f(y)$ implies that $x = y$, then we say the function is one-to-one. In this case we also write that f is $1-1$. This statement might more readily be understood using the contrapositive: If $x \neq y$ then $f(x) \neq f(y)$. The idea here is that no element of S_2 appears more than once as a the second component in any pair of a function.

If the statement that $y \in S_2$ implies that there exists an $x \in S_1$ with the property that $f(x) = y$, then we say that f is onto. Here the idea is that a function is onto if the codomain and the range are the same set. Alternatively, a function is onto if every element in the codomain appears as the second component of some pair in the function.

Let us consider some examples:

1. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither $1-1$ nor onto. It is not onto because the number -7, among others, is in the codomain but is not in the range. It is not $1-1$ because $f(2) = f(-2)$, but $2 \neq -2$.

2. Let $[-\frac{\pi}{2}, \frac{\pi}{2}]$ denote the interval on the real line containing all of the real numbers between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, including both $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Then the function $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ given by $f(x) = \sin x$ is $1-1$ but not onto. It is $1-1$, since if $x, y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ are distinct then $\sin x \neq \sin y$. But it is not onto. The range of the sine function contains all of the real

numbers between negative one and one. Therefore, there are many other real numbers that are in the codomain but are not in the range.

This examples illustrates the importance of stipulating the domain and codomain of a function before determining whether the function 1-1, onto or both. Suppose that we had taken the entire real line, instead of just the interval from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ as our domain. In this case the function would no longer be $1-1$. For then we would have $\sin \pi = \sin 3\pi = 0$. On the other hand, suppose our codomain had been defined as the interval $[-1, 1]$ rather than $\mathbb{R}$. Now the function would, indeed, be onto. The lesson here is that it is meaningless to discuss whether a function f is $1-1$ or onto, unless a domain and codomain are clearly stated.

3. In the example of the wedding guests at a reception, assume that no one sits at a table by himself and that no table remains unoccupied. Then the function that pairs a guest with the number of the table at which he will be sitting is onto but not $1-1$. It is onto since the codomain consists of the various tables, and we are assuming that no table is left unoccupied. But it is not $1-1$. Since no guest sits alone, we will have different guests being paired with the same table number.

4. Finally, the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$ is both $1-1$ and onto. To see that it is $1-1$, note that if $f(x) = f(y)$, then we must have $x + 1 = y + 1$. But this implies that $x = y$. To see that it is onto, observe that if $y \in \mathbb{Z}$, then so is $x = y - 1$. In this case we will have $f(x) = y$.

These examples show that the properties of being $1-1$ and onto are entirely independent of one another. A function might be either one without being the other. Also, a function might have both properties, or it might have neither.

Functions that are both $1-1$ and onto have a further interesting property. To see what I mean, let $S_1 = \{1, 2, 3, 4\}$ and $S_2 = \{5, 6, 7, 8\}$. Then we can define a function $f : S_1 \to S_2$ that contains the following ordered pairs:

$$(1, 5), (2, 6), (3, 7), (4, 8).$$

This function is readily seen to be both $1-1$ and onto.

As a consequence, it is possible to obtain a function going in the reverse direction from f. In other words, we can define the function $g : S_2 \to S_1$ by reversing the pairs above:

$$(5, 1), (6, 2), (7, 3), (8, 4).$$

The functions f and g bear an interesting relation to one another. Suppose we consider the composition of the two functions. In other words, first we will use f to map elements of S_1 to elements of S_2. Then we will use g to return us to S_1. We get:

$$\begin{aligned} g(f(1)) &= g(5) = 1 \\ g(f(2)) &= g(6) = 2 \\ g(f(3)) &= g(7) = 3 \\ g(f(4)) &= g(8) = 4 \end{aligned}$$

It seems the composition is the same as the function on S_1 that maps each element to itself.

Inspired by this example, we make the following definition:

Definition 3. *Let S_1 and S_2 be arbitrary sets. The **identity function** on S_1 is the function $f : S_1 \to S_1$ given by $f(x) = x$ for all $x \in S_1$.*

*Now let $f : S_1 \to S_2$ be a function that is both $1-1$ and onto. Then f is said to be an **invertible** function. The function $g : S_2 \to S_1$ with the property that the composition of f with g is the identity is said to be the **inverse** of f.*

We consider a few examples.

1. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$ is invertible. The inverse function is given by $g(y) = \frac{y-2}{3}$. You should think of the inverse of f as the function that undoes whatever f did. In this case, the function f took in a real number, multiplied it by three, and then added two. If we want to undo that, we begin by subtracting two from our number and then dividing by three.

2. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not invertible, since it is neither $1-1$ nor onto. However, if we restrict the domain and the codomain of f to the nonnegative real numbers, then f is $1-1$ and onto, and hence invertible. We find that the inverse function is given by $g(y) = \sqrt{y}$.

You might be wondering at this point why we bother to make a distinction between the codomain and the range. If any function could be made to be onto simply by restricting the codomain, why not just do it and be done with it?

The answer is that in many cases the functions of interest are defined too abstractly to determine precisely what the range is. You might be certain that your function always spits out real numbers, for example, yet not be certain precisely which real numbers they are. If you have the function whose domain is the real numbers defined as follows:

$$f(x) = \frac{\log_7(x^2(\sin x))^4}{\sqrt[7]{\tan x + e^x}} + \sin^{-1}(3x^2 + 2x),$$

then f only produces real numbers. But would you really want to have to determine the range of this function? The distinction between codomain and range is very useful in this sort of situation. Stating the codomain is the real numbers tells us something very useful. Determining the range might be a difficult or unsolvable question, however.

8.7 Problem Solving Skills

8.7.1 Proving that Function is One-to-One or Onto

If f is a function whose domain and codomain are finite and of relatively small size, it is a straightforward matter to determine whether f is $1-1$ or onto. You can simply examine a complete list of the pairs making up the function and determine "by hand," if you will, whether it satisfies some property of interest. But when your function is defined on infinite sets things are more difficult.

The most common scenario in which this occurs involves the sorts of functions you studied in your calculus classes. There you typically had functions whose domain and codomain were both the real numbers, and which were defined in terms of mathematical expressions. Establishing that functions of that sort are $1-1$ or onto requires a bit of practice.

To illustrate the basic ideas, let us consider the problem of showing that a typical linear function is both $1-1$ and onto. For concreteness, take the function $f(x) = 5x + 9$.

To prove that this function is $1-1$, we begin by reminding ourselves of the precise, mathematical definition of the term. We say a function is $1-1$

if $f(x) = f(y)$ implies that $x = y$. This, then, is the conditional proposition we must establish before we can claim success. It also suggests a very clear way to proceed.

The first line of our proof must be to assume that $f(x) = f(y)$. We will then need a chain of logical inferences that concludes with, "Therefore, $x = y$." In this case, you might already see how to continue the proof:

Proposition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 5x+9$. Then f is $1-1$.*

Proof. Assume that $f(x) = f(y)$. Then $5x + 9 = 5y + 9$. Subtracting nine from both sides gives $5x = 5y$. Dividing by five now gives $x = y$, and the proof is complete. $\square$

This sort of proof is fairly typical for problems of this sort. The first step is to assume that $f(x) = f(y)$. Then you use whatever specific information you have about the function to translate this assumption into an equation. From there you manipulate the equation as necessary.

What about showing the function is onto? Again, we begin by reminding ourselves of the precise definition of the term "onto." We need to show that if $y \in \mathbb{R}$, then there exists and $x \in \mathbb{R}$ such that $f(x) = y$. Armed with this, our way seems clear. We will begin with an arbitrary real number. We will then use what we know about the function to manufacture an x that maps to y under the function f. This leads us to the following argument:

Proposition 2. *The function $f(x) = 5x + 9$ is onto.*

Proof. Let $y \in \mathbb{R}$. Let $x = \frac{y-9}{5}$. Then

$$f(x) = f\left(\frac{y-9}{5}\right) = 5\left(\frac{y-9}{5}\right) + 9 = y.$$

This shows that y is in the range of f and the proof is complete. $\square$

These sorts of problems are a classic example of the importance of thinking clearly and with precision when crafting mathematical proofs. If you use the definition precisely as stated, the argument virtually writes itself. If you try to take a shortcut, or get by with some looser definition of the relevant terms, you will find yourself struggling.

8.7.2 Finding the Inverse of a Function

Having determined that a function is $1-1$ and onto, the next question concerns actually finding the inverse. Again, for functions on small, finite sets this problem is straightforward. One simply lists all of the pairs of the function f, and reverses them to find the inverse function g. This procedure obviously does not work for functions on infinite sets. However, there is a standard procedure for finding the inverse of a function that is described by a mathematical formula.

According to the definition of the term "inverse function," if the pair (x, y) appears in a function f, then the pair (y, x) must appear in the inverse function. A typical formula will express y in terms of x. That is, you choose some real number, plug it into the formula in place of x, and then carry out whatever evaluations are called for. So if we want to start from y and work our way back to x, it seems logical to simply solve the given equation for x.

An example should make the idea clear:

Problem 1. *Let $\mathbb{R}^+$ denote the positive real numbers. Define $f : [-\frac{2}{3}, \infty) \to \mathbb{R}^+$ by the formula*

$$f(x) = \sqrt{3x+2}.$$

Find the inverse of $f(x)$.

If we use y as a stand-in for $f(x)$, then the problem defines y by the formula $y = \sqrt{3x+2}$. Thus, given y we have a method for determining y. To obtain the value of x that produces a particular value of y, we just solve the formula for x:

$$\begin{aligned} y &= \sqrt{3x+2} \\ y^2 &= 3x+2 \\ \frac{y^2-2}{3} &= x \end{aligned}$$

This answer makes sense. The function f multiplies a number by three, adds two, and then takes a square root. If you want to undo all of that you would first square the number, subtract two and then divide by three. And that is precisely the answer we obtained.

8.8 Problems

1. Give specific examples of relations to show that the properties of reflexivity, symmetry and transitivity are logically distinct. That is, no two of them implies the third. Feel free to use examples discussed in the chapter.

2. Find the domain and range of each of the following functions:

   (a) The function that assigns to each nonnegative integer its last digit.

   (b) The function that assigns the next largest integer to a positive integer.

   (c) The function that assigns to a five-digit positive integer the number of even digits.

3. Consider the relation defined on $\mathbb{R}$ defined as follows: Two real numbers x and y are related if $xy \geq 0$. Is this relation reflexive? Symmetric? Transitive?

4. Let R be the relation defined on the integers as follows: x is related to y if $5|(x-y)$. Prove that this is an equivalence relation and describe the equivalence classes.

5. Consider the following relation on the set $\{1, 2, 3, 4, 5\}$:

$$R = \{(1,1),(1,3),(1,5),(2,2),(2,4),\\ (3,1),(3,3),(3,5),(4,2),(4,4),(5,1),(5,3),(5,5)\}.$$

   Determine all of the equivalence classes of R.

6. Find all possible partitions of the set $\{1, 2, 3, 4, 5\}$.

7. Define the following relation on the set of human beings on the planet: A and B are related if they have precisely the same parents. Prove that this is an equivalence relation and describe a typical equivalence class.

8. There is only one possible equivalence relation on a set with one element. If your set is $\{a\}$, then the only equivalence relation is (a, a). On a two element set $\{a, b\}$ you can have the equivalence relations

$(a, a), (b, b)$, and $(a, a), (a, b), (b, a), (b, b)$. How many equivalence relations are there on a three element set? How many are there on a four element set?

9. Which of the following are equivalence relations?

   (a) $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$, on the set $\{1, 2, 3\}$.

   (b) $R = \{1, 2, 3\} \times \{1, 2, 3\}$ on the set $\{1, 2, 3\}$.

   (c) $R = \{1, 2, 3\} \times \{1, 2, 3\}$ on the set $\{1, 2, 3, 4\}$.

10. Suppose that R and S are two relations on the same set. If both relations are transitive, does it follow that $R \cup S$ is also transitive? How about $R \cap S$? What if we replace the word "transitive" with the word "symmetric"? What if we use the word "reflexive"?

11. Let R be a relation on a set S. Prove that if R is symmetric and transitive, and every element of S is related to at least one other element, then R is reflexive.

12. Let $A = \{a, b, c\}$ and let $B = \{\alpha, \beta\}$. Write down all functions $f : A \to B$. Indicate which ones are $1 - 1$ or onto.

13. By a seven-bit string we mean any sequence of zeros and ones having precisely seven elements. Typical examples are 1110010 and 0101001. Define two strings to be related if their first four elements are the same. Prove that this is an equivalence relation. How many equivalence classes are there? Write down [1110001], the equivalence class containing 1110001.

14. How many functions are there from a set of size n to a set of size m? How many of these functions are $1 - 1$? How many are onto?

15. Consider the relation defined on $\mathbb{Q}$ as follows:

$$R = \{(x, y) : x, y \in \mathbb{Q}, x^2 + y^2 = 1\}.$$

    Is this a function?

16. Prove that the function $f : (-\frac{2}{3}, \infty) \to \mathbb{R}^+$ given by $f(x) = \sqrt{3x + 2}$ is $1 - 1$ and onto.

17. Let $f(x)$ be a linear function on the real numbers. That is, $f(x) = ax + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$. Prove that f is $1-1$ and onto.

18. Let A be an n-element set and let k be a positive integer with $k \leq n$. How many functions $f : A \to \{0, 1\}$ are there with exactly k elements $a \in A$ satisfying $f(a) = 1$?

19. Define two arbitrary, finite sets to be related if there is a $1-1$ and onto function between them. Prove that this is an equivalence relation and describe the equivalence classes.

20. Let S be the set of all positive integers smaller than 1000. Write down all of the reflexive functions on S.