CALCULUS CONCEPTS

1) LIMITS

$|x| = \sqrt{x^2} \geq 0$, $|x|$ is the distance x is from 0, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$|A - B|$ is the distance A is from B

$\lim_{x \to a} f(x) = L$ means if $0 < |x - a|$ is small enough then $|f(x) - L|$ is nearly 0

$\lim_{x \to a} f(x) = L$ means if $x \neq a$, but the distance x is from a is small enough then the distance $f(x)$ is from L is nearly 0.

SOME LIMIT RULES

Suppose c is a number and $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

(1) $\lim_{x \to a} c = c$

(2) $\lim_{x \to a} x = a$

(3) $\lim_{x \to a} f \pm g(x) = L \pm M$

(4) $\lim_{x \to a} cf(x) = cL$

(5) $\lim_{x \to a} fg(x) = LM$

(6) If $M \neq 0$ then $\lim_{x \to a} \frac{f}{g}(x) = \frac{L}{M}$

(7) If $M = 0$ and $L = 0$ then $\lim_{x \to a} \frac{f}{g}(x)$ DOES NOT EXIST(DNE)

(8) $\lim_{h \to 0} \frac{\sin h}{h} = 1$ (9) $\lim_{x \to 0^+} \frac{1}{x} = \infty$ (DNE)

(10) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ (DNE)

(11) $\lim_{x \to -\infty} \frac{1}{x} = 0$

(12) $\lim_{x \to \infty} \frac{1}{x} = 0$

2) CONTINUITY

The function f is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$

The function f is continuous if it is continuous at each number in its domain.

SOME CONTINUITY RULES

Suppose c is a number, the function f is continuous at $x = a$, the function g is continuous at $x = a$ and the function p is continuous at $x = f(a)$

(1) cf is continuous at $x = a$
(2) $f \pm g$ is continuous at $x = a$
(3) fg is continuous at $x = a$
(4) If $g(a) \neq 0$ then $\frac{f}{g}$ is continuous at $x = a$
(5) $p \circ f$ is continuous at $x = a$
(6) polynomials are continuous

If f is continuous on $[a,b]$ then f attains a maximum and a minimum value on $[a,b]$.

Intermediate Value Theorem: If f is continuous on $[a,b]$ and $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$ then there is a c in (a,b) so that $f(c) = y$.

3) DERIVATIVES

Let the function f be defined on $[a,b]$ then the line through $(a, f(a))$ and $(b, f(b))$ is called the secant line and its slope is given by $m = \frac{f(b) - f(a)}{b - a}$

$$\Delta f(x) = f(x+h) - f(x)$$

The difference quotient of the function f is $\frac{\Delta f}{h}(x) = \frac{\Delta f(x)}{h} = \frac{f(x+h) - f(x)}{h}$.

If $\lim_{h \to 0} \frac{\Delta f}{h}(x) = \lim_{h \to 0} \frac{\Delta f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists then we say the function f is differentiable at x and write $f'(x) = \lim_{h \to 0} \frac{\Delta f}{h}(x) = \lim_{h \to 0} \frac{\Delta f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The function f is called differentiable if it is differentiable at each x in its domain.

SOME DIFFERENTIATION RULES

Let c be a number, f, g be differentiable functions

(1) $\frac{d}{dx}c = 0$

(2) $\frac{d}{dx} f \pm g(x) = f'(x) \pm g'(x)$

(3) $\frac{d}{dx}cf(x) = cf'(x)$

(4) $\frac{d}{dx} fg(x) = f'(x)g(x) + f(x)g'(x)$

(5) $\frac{d}{dx}\frac{f}{g}(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

(6) Chain Rule: $\frac{d}{dx} f \circ g(x) = \frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

(7) $\frac{d}{dx}x^r = rx^{r-1}$

(8) $\frac{d}{dx}[g(x)]^r = r[g(x)]^{r-1} g'(x)$

(9) $\frac{d}{dx}\sin x = \cos x$

(10) $\frac{d}{dx}\cos x = -\sin x$

Differentiable functions are continuous, but a continuous function may not be differentiable.

Mean Value Theorem: If the function f is differentiable on (a,b) and continuous on $[a,b]$ then there is a number $c \in (a,b)$ so that $f'(c) = \frac{f(b) - f(a)}{b-a}$. (The slope of the secant line is equal to at least the slope of one tangent line. The secant line is parallel to a tangent line.)

If $f'(a) > 0$ then $f \uparrow$ on an interval containing a. If $f'(a) < 0$ then $f \downarrow$ on an interval containing a.

4) INTEGRALS

Let the function f be defined on $[a,b]$ then $\sum_{j=1}^{n} f(x_j^*)\Delta x_j$ is called the Riemann Sum of f on $[a,b]$ with partition $P = \{a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b\}$ and where $\Delta x_j = x_j - x_{j-1}$.

If $\lim_{\|P\|\to 0}\sum_{j=1}^{n} f(x_j^*)\Delta x_j$ where $\| P \| = \max\{\Delta x_j, j=1,...,n\}$ exists then we say f is integrable on $[a,b]$ and we write $\int_a^b f(x)dx = \lim_{\|P\|\to 0}\sum_{j=1}^{n} f(x_j^*)\Delta x_j$.

SOME INTEGRAL RULES

If f is continuous on $[a,b]$ then f is integrable on $[a,b]$.

If $F'(x) = f(x)$ on $[a,b]$ then $\int_a^b f(x)dx = F(b) - F(a)$.

If $F(x) = \int_a^x f(t)dt$ on $[a,b]$ then $F'(x) = f(x)$.

If f is continuous on $[a,b]$ then $\int_a^b f(x)dx = F(b) - F(a) = f(c)(b-a)$ for some $c \in (a,b)$.

$\int f(x)dx = F(x) + c$ if and only if $F'(x) = f(x)$.

$y'(x) = f(x, y(x));\ y(x_0) = x_0$ if and only if $y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds$ for f continuous on a region containing (x_0, y_0).

Let c be a number, f, g be integrable functions

(1) $\int_a^b c dx = cx |_a^b = c(b-a)$

(2) $\int_a^b f \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(3) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(4) Chain Rule: $\int_a^b f(g(x))g'(x)dx = f(g(x))|_a^b = \int_{g(a)}^{g(b)} f(u)du$

(5) Integration By Parts: $\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx$