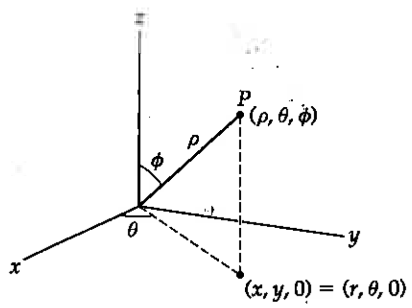
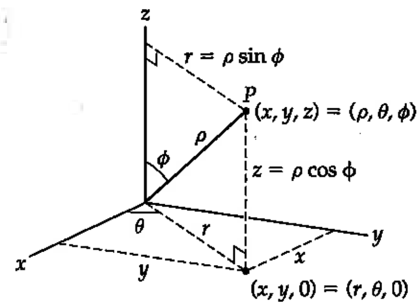
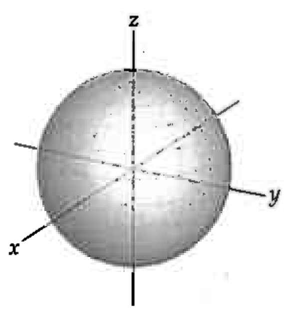
positive z-axis, as we see in the figure that follows at the left. The figure at the right shows the relationships between the spherical, cylindrical, and rectangular coordinates.



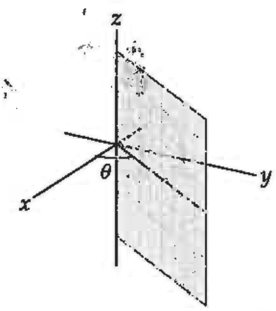


The angles θ and ϕ will be measured in radians. We will always assume that $0 \le \phi \le \pi$ and that θ lies in some interval of length 2π, typically $\theta \in [0, 2\pi]$ or $\theta \in [-\pi, \pi]$. In the figures that follow, we show a sphere, a plane, and a cone. These are the graphs obtained when ρ, θ, and ϕ, respectively, are constant.

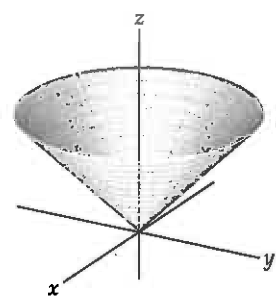
$\rho = \rho_0$



$\theta = \theta_0$



$\phi = \phi_0$



We summarize the relationships between the variables of our three-dimensional coordinate systems in the following theorem:

THEOREM 13.21

Converting Between the Three-Dimensional Coordinate Systems

Let P be a point in $\mathbb{R}^3$ with coordinates (x, y, z) in the rectangular coordinate system, (r, θ, z) in the cylindrical coordinate system, and (ρ, θ, ϕ) in the spherical coordinate system.

(a) The cylindrical coordinates and rectangular coordinates for P are related by the following equations:

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}, \quad \text{and} \quad z = z$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad z = z.$$

(b) The cylindrical coordinates and spherical coordinates for P are related by the following equations:

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \text{and} \quad \tan\phi = \frac{r}{z}$$

$$r = \rho\sin\phi, \quad \theta = \theta, \quad \text{and} \quad z = \rho\cos\phi.$$

(c) The rectangular coordinates and spherical coordinates for P are related by the following equations:

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x}, \quad \text{and} \quad \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad \text{and} \quad z = \rho\cos\phi.$$

$$V = \iiint_Q \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$