MATH 237 Sochacki Exam 2
Oct. 25, 2016 In Class Part 75 Points

Name KEY WEST
(Print) (1 point)

All necessary work must be shown for credit. Part of your score is the work you show. Work must be NEAT. You may NOT use computers, notes or texts.

I have neither received nor given help on this exam. Mon Key
(Signature) (2 points)

1. Let $\bar{v}(t) = \langle t\sin t, t\cos t\rangle$ be the velocity of a moving object with $\bar{r}(0) = \bar{0}$. Give the following. (18 points)

(a) The speed of the object.

$$\frac{ds}{dt} = \|\bar{v}(t)\| = |t|$$

(b) The path traced out by the object.

$$\bar{r}(t) = \int \bar{v}(t)\,dt$$

$$= \int \underbrace{t}_{u} \underbrace{\sin t\,dt}_{dv}\,\bar{i} + \int \underbrace{t}_{u} \underbrace{\cos t\,dt}_{dv}\,\bar{j}$$

$$= \left(-t\cos t - \int -\cos t\,dt\right)\bar{i} + \left(t\sin t - \int \sin t\,dt\right)\bar{j}$$

$$= (-t\cos t + \sin t + c_1)\bar{i} + (t\sin t + \cos t + c_2)\bar{j}$$

$$\bar{r}(0) = c_1\bar{i} + (1+c_2)\bar{j} = \bar{0} \Rightarrow c_1 = 0,\ c_2 = -1$$

$$\bar{r}(t) = \langle -t\cos t + \sin t,\ t\sin t + \cos t - 1\rangle$$

(c) The curvature of the path traced out by the object at $t = \frac{\pi}{2}$.

$$\bar{T}(t) = \frac{\bar{v}(t)}{\frac{ds}{dt}} = \frac{t\sin t\,\bar{i} + t\cos t\,\bar{j}}{t} = \sin t\,\bar{i} + \cos t\,\bar{j}$$

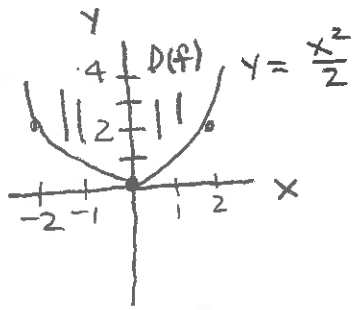
$$\bar{T}'(t) = \cos t\,\bar{i} - \sin t\,\bar{j}$$

$$\kappa = \frac{\|\bar{T}'(t)\|}{\frac{ds}{dt}} = \frac{1}{t} \qquad \kappa\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

3. Let $z = f(x,y) = \sqrt{2y - x^2}$. Give the following. (9 points)

(a) A sketch of the domain of $f$.

$$2y - x^2 \geq 0$$
$$y \geq \frac{x^2}{2}$$




(b) $z_{xx}$

$$z_x = \frac{1}{2}(2y - x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{(2y-x^2)^{1/2}} = -x(2y-x^2)^{-\frac{1}{2}}$$

$$z_{xx} = \frac{-(2y-x^2)^{1/2} + x\,\frac{1}{2}(2y-x^2)^{-\frac{1}{2}}(-2x)}{(2y-x^2)} = -(2y-x^2)^{-\frac{1}{2}} + \frac{1}{2}x(2y-x^2)^{-\frac{3}{2}}(-2x)$$

(c) $\frac{\partial^2 f}{\partial x \partial y}(1,2)$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-x}{(2y-x^2)^{1/2}}\right) = \frac{0 + x\,\frac{1}{2}(2y-x^2)^{-\frac{1}{2}}2}{(2y-x^2)} = \frac{x}{(2y-x^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial x \partial y}(1,2) = \frac{1}{\sqrt{27}}$$

4. Let $z = f(x,y) = xy + y^2$. Give the following. (9 points)

(a) The *simplified* form of $\Delta z = f(2+\Delta x, 3+\Delta y) - f(2,3)$.

$$\Delta z = (2+\Delta x)(3+\Delta y) + (3+\Delta y)^2 - 15$$
$$= 6 + 3\Delta x + 2\Delta y + \Delta x \Delta y + 9 + 6\Delta y + \Delta y^2 - 15$$
$$= 3\Delta x + 8\Delta y + \Delta x \Delta y + \Delta y^2$$

(b) The tangent plane to the graph of $f$ at the point (2,3).

$$\frac{\partial f}{\partial x}(2,3) = 3, \quad \frac{\partial f}{\partial y}(2,3) = 8$$
$$P = (2, 3, f(2,3)) = (2,3,15)$$
$$\bar{n} = \langle f_x(2,3), f_y(2,3), -1\rangle = \langle 3, 8, -1\rangle$$

$$\langle 3,8,-1\rangle \cdot \langle x-2, y-3, z-15\rangle$$
$$= 3x - 6 + 8y - 24 - z + 15 = 0$$
$$3x + 8y - z = 15$$

(c) The directional derivative of $f$ at the point (2,3) in the direction <2,-1>.

$$\nabla f(2,3) = \langle 3, 8\rangle$$

$$\bar{u} = \frac{\langle 2,-1\rangle}{\sqrt{5}} = \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right\rangle$$

$$D_{\bar{u}} f(2,3) = \nabla f(2,3) \cdot \bar{u} = \frac{6}{\sqrt{5}} - \frac{8}{\sqrt{5}} = \frac{-2}{\sqrt{5}}$$

2. Let $\bar{r}(t) = 5\cos 2t\ \bar{i} - 3\sin 2t\ \bar{j} + 4\sin 2t\ \bar{k}$. Give the following. (18 points)

(a) $\bar{T}(t), \bar{N}(t), \bar{B}(t)$

$$\bar{r}'(t) = -10\sin 2t\ \bar{i} - 6\cos 2t\ \bar{j} + 8\cos 2t\ \bar{k}$$

$$\frac{ds}{dt} = \|\bar{r}'(t)\| = \sqrt{100\sin^2 2t + 36\cos^2 2t + 64\cos^2 2t} = 10 \Rightarrow s = 10t$$

$$\bar{T}(t) = \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|} = -\sin 2t\ \bar{i} - \frac{3}{5}\cos 2t\ \bar{j} + \frac{4}{5}\cos 2t\ \bar{k}$$

$$\bar{N}(t) = \frac{\bar{T}'(t)}{\|\bar{T}'(t)\|} = \frac{-2\cos 2t\ \bar{i} + \frac{6}{5}\sin 2t\ \bar{j} - \frac{8}{5}\sin 2t\ \bar{k}}{2}$$

$$= -\cos 2t\ \bar{i} + \frac{3}{5}\sin 2t - \frac{4}{5}\sin 2t\ \bar{k}$$

$$\bar{B}(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin 2t & -\frac{3}{5}\cos 2t & \frac{4}{5}\cos 2t \\ -\cos 2t & \frac{3}{5}\sin 2t & -\frac{4}{5}\sin 2t \end{vmatrix} = \bar{i}(0) - \bar{j}\left(\frac{4}{5}\right) + \bar{k}\left(-\frac{3}{5}\right) = -\frac{4}{5}\bar{j} - \frac{3}{5}\bar{k}$$

(b) The tangential and normal components of the acceleration vector.

$$\kappa = \frac{\|\bar{T}'(t)\|}{\frac{ds}{dt}} = \frac{2}{10} = \frac{1}{5}$$

$$\frac{ds}{dt} = 10 \qquad \frac{d^2s}{dt^2} = 0$$

$$\bar{a} = 0\,\bar{T} + \left(\frac{1}{5}\right)(10)^2\,\bar{N}$$

$$a_T = 0, \quad a_N = 20$$

(c) The equation of a plane containing $\bar{r}''\left(\frac{\pi}{4}\right)$.

$\bar{a}\left(\frac{\pi}{4}\right) = \bar{r}''\left(\frac{\pi}{4}\right)$ is in the plane $\perp$ to $\bar{B}\left(\frac{\pi}{4}\right)$

$\bar{n} = B\left(\frac{\pi}{4}\right) = -\frac{4}{5}\bar{j} - \frac{3}{5}\bar{k} \qquad P = \bar{r}\left(\frac{\pi}{4}\right) = 0\bar{i} - 3\bar{j} + 4\bar{k}$

$$\left\langle 0, -\frac{4}{5}, -\frac{3}{5} \right\rangle \cdot \langle x-0, y+3, z-4 \rangle = 0$$

$$0x - \frac{4}{5}(y+3) - \frac{3}{5}(z-4) = 0$$

$$4(y+3) + 3(z-4) = 0 \Rightarrow 4y + 3z = 0$$

5. Let $w = f(x,y,z) = \ln(9 - x^2 - y^2 - z^2)$. Give the following. (18 points)

(a) A description of the domain, range and level surfaces of $f$.

$D(f)$: $9 - x^2 - y^2 - z^2 > 0$ $\quad R(f) = (-\infty, \ln 9]$

$9 > x^2 + y^2 + z^2$

Spheres with $\rho < 3$ ✓

Level Surfaces

$w = c = \ln(9 - x^2 - y^2 - z^2)$

$e^c = 9 - x^2 - y^2 - z^2$

$x^2 + y^2 + z^2 = 9 - e^c$ $\quad$ spheres $\rho = \sqrt{9 - e^c}$

(b) $D_{\bar{u}} f(1,1,1)$ for $\bar{u}$ the direction of maximum increase in $f$ at the point (1,1,1).

$$\nabla f(x,y,z) = \left\langle \frac{-2x}{9-x^2-y^2-z^2}, \frac{-2y}{9-x^2-y^2-z^2}, \frac{-2z}{9-x^2-y^2-z^2} \right\rangle$$

$$\nabla f(1,1,1) = \left\langle \frac{-2}{6}, \frac{-2}{6}, \frac{-2}{6} \right\rangle = \left\langle \frac{-1}{3}, \frac{-1}{3}, \frac{-1}{3} \right\rangle$$

$$\bar{u} = \frac{\nabla f(1,1,1)}{\|\nabla f(1,1,1)\|}$$

$$D_{\bar{u}} f(1,1,1) = \nabla f(1,1,1) \cdot \frac{\nabla f(1,1,1)}{\|\nabla f(1,1,1)\|} = \|\nabla f(1,1,1)\| = \sqrt{3\left(\tfrac{1}{9}\right)} = \sqrt{\tfrac{1}{3}}$$

(c). Give the equation of a tangent plane for the level surface $w = \ln(5)$.

$w = \ln 5 = \ln(9 - x^2 - y^2 - z^2)$ $\quad x = 2,\ y = 0,\ z = 0$

$$\bar{n} = \nabla f(2,0,0) = \left\langle \frac{-4}{5}, 0, 0 \right\rangle$$

$$\left\langle \frac{-4}{5}, 0, 0 \right\rangle \cdot \langle x-2, y-0, z-0 \rangle = 0$$

$$-\frac{4}{5}(x-2) = 0$$

$$x = 2$$