MATH 237 Sochacki Home Work 3 Name KEY WEST
Nov. 11, 2016 Take Home 50 Points (Print) (1 point)

All necessary work must be shown for credit. Your work must represent the question asked. You may NOT use computers. You may use your notes or text. Your work must be neat or I will not grade your work. You may discuss this assignment with others, but all work turned in must be your own work.

I have neither received nor given help on this exam. Don Key
(Signature) (1 point)

1. Give the average value of $z = f(x,y) = x^2 - y^2$ over the region $Q = \{(x,y) \mid -2 \le x \le 2, -2 \le y \le 2\}$. (6 points)

$$\frac{\int_{-2}^{2}\int_{-2}^{2}(x^2-y^2)\,dx\,dy}{16} = \frac{1}{16}\int_{-2}^{2} \frac{x^3}{3} - xy^2 \Big|_{-2}^{2} dy$$

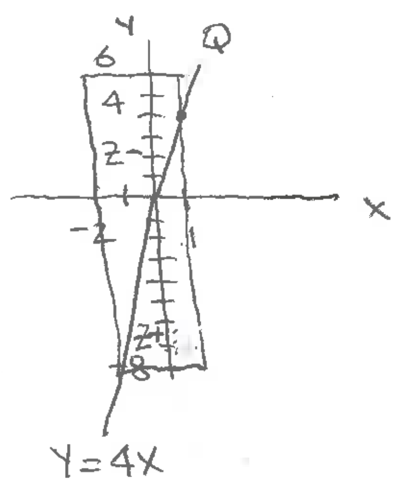
$$= \frac{1}{16}\int_{-2}^{2}\left(\frac{8}{3}-2y^2\right)-\left(-\frac{8}{3}+2y^2\right) dy$$

$$= \frac{1}{16}\int_{-2}^{2}\left(\frac{16}{3}-4y^2\right) dy$$

$$= \frac{1}{16}\left(\frac{16}{3}y - \frac{4}{3}y^3\right)\Big|_{-2}^{2} = \frac{1}{16}(0) = 0$$

2. Give $\int_{-8}^{6}\int_{-2}^{1}|4x - y|\,dx\,dy$. (6 points)

Let $z = 4x - y \Rightarrow z = 0 \Rightarrow y = 4x$



$$= -\int_{-2}^{1}\int_{4x}^{6}(4x-y)\,dy\,dx + \int_{-2}^{1}\int_{-8}^{4x}(4x-y)\,dy\,dx$$

$$= \int_{-2}^{1}\int_{4x}^{6}(y-4x)\,dy\,dx + \int_{-2}^{1}\int_{-8}^{4x}(4x-y)\,dy\,dx$$

$$= \int_{-2}^{1} \frac{y^2}{2} - 4xy\Big|_{4x}^{6} dx + \int_{-2}^{1} 4xy - \frac{y^2}{2}\Big|_{-8}^{4x} dx$$

$$= \int_{-2}^{1}(18-24x-8x^2+16x^2)\,dx + \int_{-2}^{1}(16x^2-8x^2+32x+32)\,dx$$

$$= \left(18x - 12x^2 + \frac{8}{3}x^3\right)\Big|_{-2}^{1} + \left(\frac{8}{3}x^3 + 16x^2 + 32x\right)\Big|_{-2}^{1}$$

*

$$= \left(\left(18-12+\frac{8}{3}\right)-\left(-36-48-\frac{64}{3}\right)\right) + \left(\frac{8}{3}+16+32\right)-\left(-\frac{64}{3}+64-64\right)$$

$$= 6 + \frac{144}{3} + 84 + 48 = \underline{186}$$

3. Let $z = x^2 + 2xy + y^2$. Determine where $\nabla z = \bar{0}$. Determine the level curves of z. Show ∇z is orthogonal to the level curves of z. (Hint: Parametrize the level curves of z.) Give the tangent plane and the line normal to the tangent plane if (a) $x = 0, y = 0$, (b) $x = 1, y = 2$. (6 points)

$$\nabla z = \langle 2x+2y, 2x+2y \rangle = \langle 0,0 \rangle \Rightarrow 2x+2y=0,\ 2x+2y=0$$
$$x+y=0 \Rightarrow y=-x$$

$$z = x^2+2xy+y^2 = c$$
$$(x+y)^2 = k^2$$
$$(x+y) = \pm k$$
$$y = -x \pm k$$

level curves are parallel lines

$$\bar{r}(x) = x\bar{i} + y\bar{j} = x\bar{i} + (-x \pm k)\bar{j}$$
$$\bar{r}'(x) = \bar{i} - \bar{j}$$
$$\nabla z \cdot \bar{r}'(x) = \langle 2x+2y, 2x+2y \rangle \cdot \langle 1,-1 \rangle$$
$$= 2x+2y-(2x+2y)$$
$$= 0$$

(a) $x=0, y=0, z=0$

$$\bar{n} = \langle 2x+2y, 2x+2y, -1 \rangle$$
$$= \langle 0,0,-1 \rangle$$
$$\bar{n} \cdot \langle x-0, y-0, z-0 \rangle = -z = 0$$
$$\ell(t) = (0,0,0) + t(0,0,-1)$$
$$= -t\bar{k}$$

(b) $x=1, y=2, z=9$

$$\bar{n} = \langle 6,6,-1 \rangle$$
$$\bar{n} \cdot \langle x-1, y-2, z-9 \rangle =$$
$$6(x-1)+6(y-2)-(z-9)=0$$
$$6x+6y-z=9$$
$$\ell(t) = (1,2,9) + t(6,6,-1)$$

4. Let $w = e^{x^2+y^2-z^2}$. Describe what the level surfaces are. Give the equation of the tangent plane and the line normal to the tangent plane at the point (3,4,5). What is the level surface for this point? (6 points)

$$w = e^{x^2+y^2-z^2} = e^c$$
$$x^2+y^2-z^2 = c$$
$$z^2 = x^2+y^2-c$$

level surfaces are cones

$$\nabla w = \langle 2xe^{x^2+y^2-z^2}, 2ye^{x^2+y^2-z^2}, -2ze^{x^2+y^2-z^2} \rangle$$

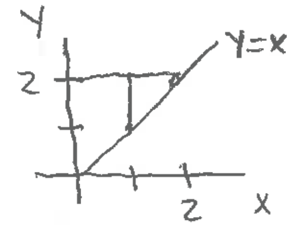
$$\nabla w(3,4,5) = \langle 6,8,-10 \rangle \qquad n = \langle 6,8,-10 \rangle$$

Plane: $\langle 6,8,-10 \rangle \cdot \langle x-3, y-4, z-5 \rangle$
$$6x-18+8y-32-10z+50=0$$
$$6x+8y-10z=0$$

$$w(3,4,5) = e^0 = e^{x^2+y^2-z^2}$$
$$z^2 = x^2+y^2 \text{ cone}$$

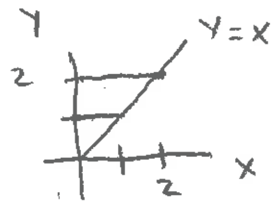
Line: $(3,4,5) + t\langle 6,8,-10 \rangle$

5. Evaluate $\int_0^2\int_x^2 y^2e^{xy}\,dy\,dx$. (6 points)



$$\int_0^2\int_x^2 \underbrace{y^2}_{u}\,\underbrace{e^{xy}\,dy}_{dv}\,dx$$

$$= \int_0^2 \left(y^2\,\frac{1}{x}\,e^{xy}\Big|_x^2 - \int_x^2 \frac{1}{x}\,e^{xy}\,2y\,dy\right)dx \quad \text{UGH!}$$



$$\int_0^2\int_0^y y^2e^{xy}\,dx\,dy = \int_0^2 y\int_0^y e^{xy}\,y\,dx\,dy$$

$$= \int_0^2 y\,e^{xy}\Big|_0^y\,dy = \int_0^2 \left(ye^{y^2} - y\right)dy$$

$$= \frac{1}{2}\int_0^2 e^{y^2}\,2y\,dy - \int_0^2 y\,dy = \frac{1}{2}e^{y^2}\Big|_0^2 - \frac{y^2}{2}\Big|_0^2$$

$$= \frac{1}{2}e^4 - \frac{1}{2} - 2 = \frac{1}{2}e^4 - \frac{5}{2}$$

6. Give the average value of $z = f(x,y) = xy + y^2$ over the region $Q = \{(x,y)\,|\,x+4 \le y \le 5\sqrt{x}\}$. (6 points)

$x+4 = 5\sqrt{x}$

$(x+4)^2 = 25x$

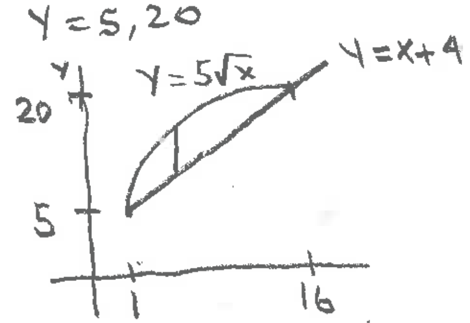
$x^2+8x+16 = 25x$

$x^2-17x+16 = 0$

$(x-1)(x-16) = 0$

$x = 1, 16$

$y = 5, 20$



$$\text{Avg} = \frac{\int_1^{16}\int_{x+4}^{5\sqrt{x}}(xy+y^2)\,dy\,dx}{\int_1^{16}\int_{x+4}^{5\sqrt{x}}dy\,dx}$$

$$= \frac{\int_1^{16} x\frac{y^2}{2} + \frac{y^3}{3}\Big|_{x+4}^{5\sqrt{x}}dx}{\int_1^{16} y\Big|_{x+4}^{5\sqrt{x}}dx}$$

$$= \frac{\int_1^{16}\left(\frac{25}{2}x^2 + \frac{125}{3}x^{\frac{3}{2}}\right) - \left(\frac{1}{2}\underbrace{x}_{u}\underbrace{(x+4)^2}_{dv} + \frac{1}{3}(x+4)^3\right)dx}{\int_1^{16}\left(5x^{\frac{1}{2}} - (x+4)\right)dx}$$

$$= \frac{\frac{25}{6}x^3 + \frac{50}{3}x^{\frac{5}{2}}\Big|_1^{16} - \frac{1}{2}\left(x\,\frac{(x+4)^3}{3}\Big|_1^{16} - \int_1^{16}\frac{(x+4)^3}{3}dx\right) - \frac{(x+4)^4}{12}\Big|_1^{16}}{\frac{10}{3}x^{\frac{3}{2}} - \frac{(x+4)^2}{2}\Big|_1^{16}}$$

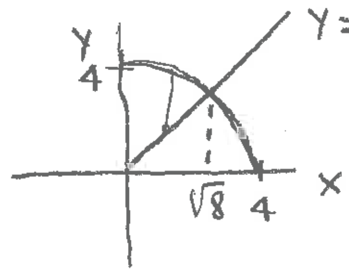
$$= \frac{\frac{68225}{2} - \frac{1}{2}\left(42625 - \frac{(x+4)^4}{12}\Big|_1^{16}\right) - \frac{53125}{4}}{\frac{640}{3} - \frac{1145}{6}}$$

$$= \frac{1095}{4} = 273.75$$

7. Give the average value of $z = \dfrac{\sqrt{x^2+y^2}}{1+\left(x^2+y^2\right)^{3/2}}$ over the region $Q = \left\{(x,y) \mid 0 \le x \le \sqrt{8}, x \le y \le \sqrt{16-x^2}\right\}$.

(6 points)

$r = \sqrt{x^2+y^2}$ $\qquad y = \sqrt{16-x^2} \Rightarrow y^2 = 16-x^2 \Rightarrow x^2+y^2 = 16$



$y = x, \ \theta = \frac{\pi}{4}$

$$\int_{\pi/4}^{\pi/2} \int_0^4 \frac{r}{1+r^3}\, r\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \int_0^4 \frac{r^2}{1+r^3}\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{3} \int_0^4 \frac{3r^2}{1+r^3}\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{3} \ln(1+r^3)\Big|_0^4 \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{3} \ln(65)\, d\theta$$

$$= \frac{\pi}{4} \frac{1}{3} \ln(65) = \frac{\pi}{12} \ln(65)$$

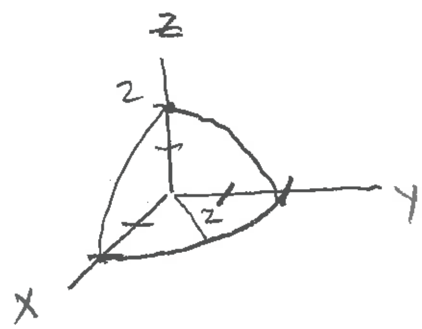
$$\text{Avg. Val.} = \frac{\frac{\pi}{12}\ln(65)}{\frac{\pi 4^2}{8}}$$

$$= \frac{1}{24} \ln 65$$

8. Give the volume of $Q = \left\{(x,y,z) \mid 0 \le z \le \sqrt{4-x^2-y^2}\right\}$. (6 points)

$z = \sqrt{4-x^2-y^2} \Rightarrow z^2 = 4-x^2-y^2 \Rightarrow x^2+y^2+z^2 = 4$

Sphere radius 2



$$V = \int_0^{2\pi} \int_0^2 \sqrt{4-r^2}\; r\, dr\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2} \int_0^2 (4-r^2)^{1/2}\, (-2r)\, dr\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\, \frac{2}{3}(4-r^2)^{3/2}\Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} 4^{3/2}\, d\theta = \frac{2\pi}{3} \cdot 8 = \frac{16}{3}\pi$$