Determinant Rules and Equivalence

Let $n \in \mathbb{N}$, $A, B \in \mathbb{M}_n$

Theorems, Corollaries, Rules

1. $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(\tilde{A}_{i,j})$ for $i \in \{1, 2, ..., n\}$.

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3. If A has a row of zeros or column of zeros then $\det(A) = 0$.

4. If A is diagonal, upper triangular or lower triangular then the $\det(A)$ is the product of the diagonals of A.

5. $\det(A) = \det(A^T)$.

6. Row operations on A:

   a. If B is formed from A by adding a multiple of one row of A to another row of A then $\det(A) = \det(B)$.

   b. If B is formed from A by switching two rows of A then $\det(A) = -\det(B)$.

   c. If B is formed from A by multiplying a row of A by a scalar c then $\det(A) = c \det(B)$ (cautionary rule).

7. A is invertible if and only if $\det(A) \neq 0$.

8. $\det(AB) = \det(A)\det(B)$. ($\det(A^{-1}) = \frac{1}{\det(A)}$).

9. (Cramer's Rule)
   Suppose that $\det A \neq 0$, $\bar{b} \in \mathbb{R}^n$ and B_j is the matrix formed by replacing the j^{th} column of A by $\bar{b}$ then the solution to the SLE $A\bar{x} = \bar{b}$ is given by

$$x_j = \frac{\det(B_j)}{\det(A)} \text{ for } j = 1, 2, ..., n.$$

The following are equivalent (all true or all false):

1. The SLE $A\bar{x} = \bar{b}$ has one and only one solution.

2. Gaussian elimination gives an upper triangular matrix for A with no zeros on the diagonal.

3. The SLE $A\bar{x} = \bar{0}$ has one and only one solution.

4. A is invertible.

5. $\det(A) \neq 0$.

6. Cramer's Rule is true.