MATH 238 Sochacki Exam 2
Mar. 29, 2021 100 Points

Name KEY WEST
(Print) (2 points)

All necessary work must be shown for credit and must represent the question asked. Your work MUST be NEAT and in PENCIL. You may NOT use computers, notes or texts. Calculators can be used only to help with arithmetic. Does not exist or no solution is always a possible answer.

I have neither received nor given help on this exam. Don Key
(Signature) (2 points)

1. Show whether or not the solutions to the first order linear homogeneous ordinary differential equation form a vector space or not. (12 points)

$y' + p(x)y = 0$

$y(x) = 0$ is a soln

Let u, v be solns

$u' + p(x)u = 0$

$v' + p(x)v = 0$

$(u' + v') + (p(x)u + p(x)v) = 0$

$(u + v)' + p(x)(u+v) = 0$

$u+v$ is a soln

closed under vector addition

$c(u' + p(x)u) = c \cdot 0 = 0$

$cu' + p(x)cu = 0$

$(cu)' + p(x)(cu) = 0$

cu is a soln

closed under scalar mult

is a v.s.

2. Give at least two solutions to $y' = y^{\frac{2}{5}};\ y(0) = 0$. (12 points)

$y(x) = 0 \Rightarrow y' = 0 = 0^{\frac{2}{5}}$ (1)

$y(0) = 0$

$y^{-\frac{2}{5}} y' = 1$

$\frac{5}{3} y^{\frac{3}{5}} = x + c$

$y^{\frac{3}{5}} = \frac{3}{5}x + c$

$y = \left(\frac{3}{5}x + c\right)^{\frac{5}{3}}$

$y(0) = c^{\frac{5}{3}} = 0 \Rightarrow c = 0$

$y = \left(\frac{3}{5}x\right)^{\frac{5}{3}}$ (2)

3. Show whether or not $10x - 4x^2$ is in the span of $S = \{1 - 2x + x^2, 2 + x, 1 + 3x - x^2\}$. Also show if S is linearly independent or dependent. (12 points)

$$10x - 4x^2 = c_1(1 - 2x + x^2) + c_2(2 + x) + c_3(1 + 3x - x^2)$$

$$0 = c_1 + 2c_2 + c_3$$
$$10 = -2c_1 + c_2 + 3c_3$$
$$-4 = c_1 + 0c_2 - c_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & 1 & 3 & 10 \\ 1 & 0 & -1 & -4 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 10 \\ 0 & -2 & -2 & -4 \end{pmatrix} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$c_3 = 0$
$c_2 = 2$
$c_1 = -2c_2 - c_3$
$= -4$

$-4(1 - 2x + x^2) + 2(2 + x) = 10x - 4x^2$

is in span

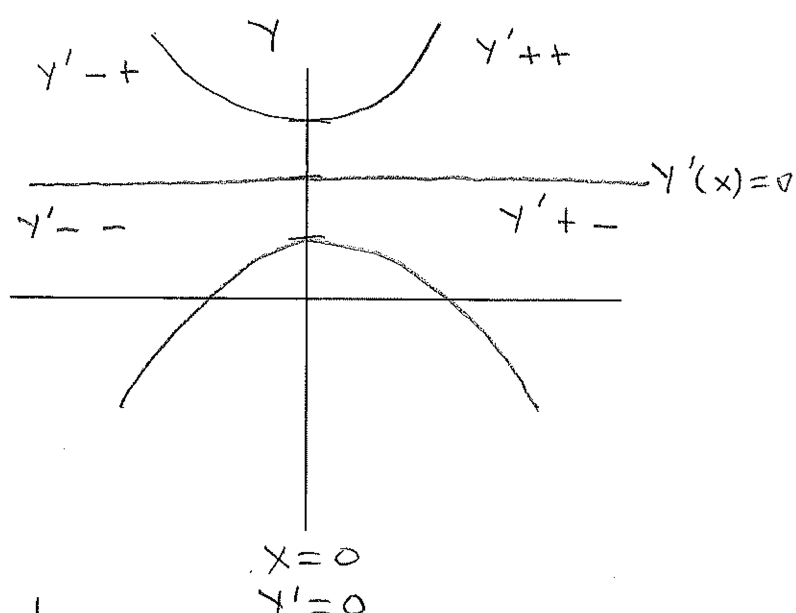
S is lin. dep. since there are inf. many solns to H problem

4. Sketch a phase portrait with enough solutions using the equilibrium solutions and concavity for the ordinary differential equation $\frac{dy}{dx} = 2xy - 4x$ and give the solution satisfying $y(0) = 2$. (12 points)

$$\frac{dy}{dx} = 2x(y - 2)$$

$y(x) = 2$ EQ soln

$$y'' = 2(y - 2) + 2xy'$$
$$= 2(y-2) + 2x\,2x(y-2)$$
$$= 2(y-2)\underbrace{(1 + 4x^2)}_{+}$$



$y(x) = 2$ is a soln and satisfies $y(0) = 2$
It is the only soln that does this.

5. Let $A = \begin{pmatrix} -3 & 9 & -9 \\ 1 & -3 & 3 \\ 2 & -6 & 6 \end{pmatrix}$. Give dim(RS(A)) and dim(CS(A)) and a basis for the NS(A). (24 points)

RS(A)

$$\begin{pmatrix} 1 & -3 & 3 \\ -3 & 9 & -9 \\ 2 & -6 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\dim(RS(A)) = 1$

CS(A) $= RS(A^T)$

$$A^T = \begin{pmatrix} -3 & 1 & 2 \\ 9 & -3 & -6 \\ -9 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\dim(CS(A)) = 1$

NS(A)

$$\left(\begin{array}{ccc|c} -3 & 9 & -9 & 0 \\ 1 & -3 & 3 & 0 \\ 2 & -6 & 6 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ -3 & 9 & -9 & 0 \\ 2 & -6 & 6 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - 3x_2 + 3x_3 = 0$$

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ are two lin ind solns and therefore a basis for NS(A).

6. Solve $x^3y' + 2x^2y = 6x^2 \sin x$. Write the answer as a linear combination of the solution to the homogeneous problem and the nonhomogeneous problem. (24 points)

$$y' + \frac{2}{x} y = \frac{6}{x} \sin x$$

$$U(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2$$

$$(x^2 y)' = 6x \sin x$$

$$x^2 y = \int \underbrace{6x}_{u} \underbrace{\sin x \, dx}_{dv}$$

$$\int = -6x \cos x - \int -\cos x \; 6 \, dx$$

$$= -6x \cos x + 6 \int \cos x \, dx$$

$$x^2 y = -6x \cos x + 6 \sin x + C$$

$$y = \underbrace{-\frac{6}{x} \cos x + \frac{6}{x^2} \sin x}_{NH} + c \underbrace{\left(\frac{1}{x^2}\right)}_{H}$$

$$y_H = \frac{1}{x^2} = x^{-2}$$

$$y_H' + \frac{2}{x} y_H = -2x^{-3} + \frac{2}{x} x^{-2} = 0$$