MATH 238 Sochacki Exam 2
Mar. 29, 2021 100 Points

Name KEY WEST
(Print) (2 points)

All necessary work must be shown for credit and must represent the question asked. Your work MUST be NEAT and in PENCIL or computer touch screen. You may NOT use computers, notes or texts. Calculators can be used only to help with arithmetic. Does not exist or no solution is always a possible answer.

I have neither received nor given help on this exam. ______________________
(Signature) (2 points)

1. Give the solution to $y'' + y' - 2y = 0; y(1) = 1, y'(1) = 1$. (12 points)

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$Y_H = c_1 e^{-2x} + c_2 e^{x}$

$Y_H' = -2c_1 e^{-2x} + c_2 e^{x}$

$Y_H(1) = c_1 e^{-2} + c_2 e = 1$

$Y_H'(1) = -2c_1 e^{-2} + c_2 e = 1$

$3c_1 e^{-2} = 0$

$c_1 = 0$

$c_2 = \frac{1}{e}$

$Y_H = \frac{1}{e} e^{x} = e^{x-1}$

2. Give the eigenvalues and of eigenvectors for $A = \begin{pmatrix} -8 & 5 \\ -13 & 8 \end{pmatrix}$. (12 points)

$\det(A - \lambda I) = \begin{vmatrix} -8-\lambda & 5 \\ -13 & 8-\lambda \end{vmatrix} = \lambda^2 - 64 + 65 = \lambda^2 + 1 = 0$

$\lambda = \pm i$

$\underline{\lambda = i}$ $\begin{pmatrix} -8-i & 5 \\ -13 & 8-i \end{pmatrix} \Rightarrow \begin{pmatrix} -13 & 8-i \\ -8-i & 5 \end{pmatrix} \xrightarrow{\frac{8+i}{-13}} \begin{pmatrix} -13 & 8-i \\ 0 & 0 \end{pmatrix}$

$-13v_1 + (8-i)v_2 = 0$ $\quad v_2 = 13 \quad v_1 = 8-i \quad \vec{v} = \begin{pmatrix} 8-i \\ 13 \end{pmatrix}$

$\underline{\lambda = -i}$ $\quad \bar{v} = \begin{pmatrix} 8+i \\ 13 \end{pmatrix}$

3. Give the general solution to $y''' + 8y' = 1 - e^{-2x} + 4\cos 8x$. (Do NOT solve for any coefficients.) (12 points)

$r^3 + 8r = 0$

$r(r^2+8) = 0$

$r = 0, \ \pm i\sqrt{8} = \pm 2\sqrt{2}\, i$

$y = c_1 + c_2 \cos 2\sqrt{2}\, x + c_3 \sin 2\sqrt{2}\, x + Ax + B + Ce^{-2x}$

$+ D\cos 8x + E\sin 8x$

4. Give a differential equation whose characteristic equation is $(r-2)(r^2+4)$ and give the homogeneous solution to this differential equation. (12 points)

$(r-2)(r^2+4) = r^3 + 4r - 2r^2 - 8 = r^3 - 2r^2 + 4r - 8$

$y''' - 2y'' + 4y' - 8y = 0 \qquad r = 2, \pm 2i$

$y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$

5. Give the eigenvalues and eigenvectors for A, A^T, A^{-1} and a diagonal matrix A is similar to for $A = \begin{pmatrix} -5 & 4 & 9 \\ -6 & 6 & 18 \\ 0 & 0 & -2 \end{pmatrix}$. (24 points)

$$\det(A-\lambda I) = \begin{vmatrix} -5-\lambda & 4 & 9 \\ -6 & 6-\lambda & 18 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -5-\lambda & 4 \\ -6 & 6-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(\lambda^2-\lambda-30+24)$$

$$= (-2-\lambda)(\lambda^2-\lambda-6)$$

$$= -(\lambda+2)(\lambda+2)(\lambda-3)$$

A is INV.

$\underline{\lambda=-2}$

$$\begin{pmatrix} -3 & 4 & 9 \\ -6 & 8 & 18 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & 4 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-3V_1 + 4V_2 + 9V_3 = 0$$

$$\bar{V} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \bar{V} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$$

A is SIMILAR TO $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$\underline{\lambda=3}$

$$\begin{pmatrix} -8 & 4 & 9 \\ -6 & 3 & 18 \\ 0 & 0 & -5 \end{pmatrix} \Rightarrow \begin{pmatrix} -8 & 4 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$V_3 = 0 \quad -2V_1 + V_2 = 0 \quad V_2 = 2V_1$$

$$\bar{V} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

A^{-1}

$$\lambda = \frac{1}{-2} \quad \bar{V} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$$

$$\lambda = \frac{1}{3} \quad \bar{V} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} -5 & -6 & 0 \\ 4 & 6 & 0 \\ 9 & 18 & -2 \end{pmatrix}$$

$\underline{\lambda=-2}$

$$\begin{pmatrix} -3 & -6 & 0 \\ 4 & 8 & 0 \\ 9 & 18 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 8 & 0 \\ 9 & 18 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V_1 + 2V_2 = 0$$

$$\bar{V} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$\underline{\lambda=3}$

$$\begin{pmatrix} -8 & -6 & 0 \\ 4 & 3 & 0 \\ 9 & 18 & -5 \end{pmatrix}$$

$$\begin{pmatrix} -8 & -6 & 0 \\ 0 & 0 & 0 \\ 1 & 12 & -5 \end{pmatrix} \quad \begin{matrix} -8V_1 - 6V_2 = 0 \\ V_1 + 12V_2 - 5V_3 = 0 \end{matrix}$$

$$\bar{V} = \begin{pmatrix} 6 \\ -8 \\ -18 \end{pmatrix}$$

6. Give the general solution to $x^2y'' - 2y = 3x^2$. (8 points)

$y_1 = x^2,\ y_1' = 2x,\ y_1'' = 2 \qquad x^2 y_1'' - 2y = x^2 2 - 2x^2 = 0$

$y_2 = uy_1 = ux^2,\ y_2' = u'x^2 + 2xu,\ y_2'' = u''x^2 + u'2x + u'2x + 2u$

$= u''x^2 + 4xu' + 2u$

$x^2 y_2'' - 2y_2 = x^2(u''x^2 + 4xu' + 2u) - 2ux^2$

$= x^2(u''x^2 + 4xu') = 0$

$\to u''x^2 + 4xu' = 0 \quad v = u'$

$v'x + 4v = 0$

$\frac{v'}{v} = \frac{-4}{x}$

$\ln v = -4\ln x$

$v = x^{-4}$

$u = \frac{x^{-3}}{-3}$

$\therefore y_2 = ux^2 = x^{-3}x^2 = x^{-1}$

$y_1 = x^2,\ y_2 = x^{-1}$

$W(x^2, x^{-1}) = \left|\begin{pmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{pmatrix}\right| = -3$

$u_1 = \int \frac{\begin{vmatrix} 0 & x^{-1} \\ 3 & -x^{-2} \end{vmatrix}}{-3} dx = \int \frac{-3x^{-1}}{-3} dx = \ln|x|$

$u_2 = \int \frac{\begin{vmatrix} x^2 & 0 \\ 2x & 3 \end{vmatrix}}{-3} dx = \frac{-x^3}{3}$

$y = c_1x^2 + c_2x^{-1} + x^2\ln|x| - \frac{x^3}{3}x^{-1}$

7. Let $A \in M_2$. Show that if $\lambda_1 \neq \lambda_2$ are the eigenvalues of A then the eigenvectors of A are linearly independent. (8 points)

$A\bar{v}_1 = \lambda_1 \bar{v}_1 \quad A\bar{v}_2 = \lambda_2\bar{v}_2$

If $c_1\bar{v}_1 + c_2\bar{v}_2 = \bar{0}$ then $A(c_1\bar{v}_1 + c_2\bar{v}_2) = A\bar{0} = \bar{0}$

$c_1A\bar{v}_1 + c_2A\bar{v}_2 = \bar{0}$

$c_1\lambda_1\bar{v}_1 + c_2\lambda_2\bar{v}_2 = \bar{0}$

$-\lambda_1 \quad c_1\bar{v}_1 + c_2\bar{v}_2 = \bar{0}$

$(c_2\lambda_2 - c_2\lambda_1)\bar{v}_2 = \bar{0}$

$c_2(\lambda_2 - \lambda_1)\bar{v}_2 = 0$

$c_2 = 0,\ \lambda_2 \neq \lambda_1,\ \bar{v}_2 \neq \bar{0}$

$c_1 = 0,\ \bar{v}_1 \neq 0$ lin. ind.

8. Convert $y'' - 2cy' + c^2y = 4e^{cx}$ for c a real number to a matrix system of ODEs. Be sure to indicate what A is. (8 points)

$v = y'$

$y' = v$

$v' = y'' = -c^2y + 2cy' + 4e^{cx} = -c^2y + 2cv + 4e^{cx}$

$$\begin{pmatrix} y \\ v \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -c^2 & 2c \end{pmatrix}}_{A}\begin{pmatrix} y \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 4e^{cx} \end{pmatrix}$$