MATH 238 Sochacki Exam 3
June 6, 2014 In Class Part 150 Points

Name KEY WEST
(Print) (3 points)

All necessary work must be shown for credit and must represent the question asked. Your work MUST be NEAT. You may NOT use computers, notes or texts. Calculators can be used only to help with arithmetic.

I have neither received nor given help on this exam. Don Key
(Signature) (3 points)

1. Solve $y'=1+y^2; y(0)=1$ (12 points)

$$\frac{dy}{dx} = 1+y^2$$

$$\frac{dy}{1+y^2} = dx$$

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1} y = x + c$$

$$y = \tan(x+c)$$

$$y(0) = \tan c = 1 \qquad c = \frac{\pi}{4}$$

$$y = \tan\left(x + \frac{\pi}{4}\right)$$

2. Solve $\frac{dy}{dx} = \frac{\cos y - 2 - 2ye^{2x}}{e^{2x} + x\sin y}; y(0)=0$. (12 points)

$$(e^{2x} + x\sin y)\,dy = (\cos y - 2 - 2ye^{2x})\,dx$$

$$(2ye^{2x} + 2 - \cos y)\,dx + (e^{2x} + x\sin y)\,dy = 0$$

$$F(x,y) = \int (e^{2x} + x\sin y)\,dy = ye^{2x} - x\cos y + h(x)$$

$$\frac{\partial F}{\partial x} = 2ye^{2x} - \cos y + h'(x) = 2ye^{2x} + 2 - \cos y$$

$$h'(x) = 2 \qquad h(x) = 2x$$

$$F(x,y) = ye^{2x} - x\cos y + 2x$$

$$F(0,0) = 0 = c \qquad y\,e^{2x} - x\cos y + 2x = 0$$

3. Solve $y' = 2y + x; y(1) = 1$. (12 points)

$$y' - 2y = x$$

$$p(x) = -2 \quad u(x) = e^{\int -2\,dx} = e^{-2x}$$

$$e^{-2x}y' - 2e^{-2x} = xe^{-2x}$$

$$\frac{d}{dx}(e^{-2x}y) = xe^{-2x}$$

$$e^{-2x}y = \int \underbrace{x}_{u}\underbrace{e^{-2x}\,dx}_{dv} = \frac{-x}{2}e^{-2x} - \int \frac{-1}{2}e^{-2x}\,dx$$

$$= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$

$$y = \frac{-x}{2} - \frac{1}{4} + ce^{2x}$$

$$y(1) = \frac{-1}{2} - \frac{1}{4} + ce^{2} = 1 \qquad c = \frac{7}{4}e^{-2}$$

$$y = \frac{-x}{2} - \frac{1}{4} + \frac{7}{4}e^{-2}e^{2x}$$

4. Solve $y'' + 4y' = 0$. (12 points)

$$v = y'$$

$$v' + 4v = 0$$

$$v' = -4v \Rightarrow v = ce^{-4x}$$

OR

$$\frac{dv}{dx} = -4v \qquad y' = ce^{-4x}$$

$$\frac{1}{v}dv = -4\,dx \qquad y = -\frac{c}{4}e^{-4x} + c_2$$

$$\ln v = -4x + c \qquad = c_1e^{-4x} + c_2$$

$$v = ce^{-4x}$$

5. Solve $y'' = 2yy'; y(1) = 0, y'(1) = 1$. (12 points)

$V = y'$

$$\frac{dv}{dy} = \frac{v'}{y'} = \frac{y''}{y'} = \frac{2yy'}{y'} = 2y$$

$$dv = 2y\,dy$$

$$v = y^2 + c$$

$$y' = y^2 + c$$

$$y'(1) = y(1)^2 + c$$

$$1 = 0 + c$$

$$y' = y^2 + 1$$

$$\frac{dy}{dx} = y^2 + 1$$

$$\frac{dy}{y^2+1} = dx$$

$$\tan^{-1} y = x + c$$

$$\tan^{-1}(0) = 1 + c \Rightarrow c = -1$$

$$y = \tan(x-1)$$

6. Show $\{e^{-x}\sin 2x, e^{-x}\cos 2x\}$ is a linearly independent set. (12 points)

$$W = \begin{vmatrix} e^{-x}\sin 2x & e^{-x}\cos 2x \\ -e^{-x}\sin 2x + 2e^{-x}\cos 2x & -e^{-x}\cos 2x - 2e^{-x}\sin 2x \end{vmatrix}$$

$$= -2e^{-2x}\sin^2 2x - 2e^{-2x}\cos^2 2x = -2e^{-2x} \neq 0$$

7. Consider the first order ODE $y' = x\cos y$. Give the equilibrium solutions of this ODE. Give a direction field for the points $(-1,-\pi),(-1,0),(-1,\pi),(0,0),(1,-\pi),(1,0),(1,\pi)$. Using the first and second derivative, the equilibrium solutions and your direction field sketch a phase portrait with at least 8 possible solutions. (24 points)

$y'' = \cos y - x\sin y\, y' = \cos y - x\sin y\, x\cos y = \cos y(1-x^2\sin y)$

EQ: $y' = 0 \Rightarrow x = 0$ or $\cos y = 0 \Rightarrow y = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$

$y'(-1,-\pi) = -\cos(-\pi) = 1$

$y'(-1,0) = -\cos 0 = -1$

$y'(-1,\pi) = -\cos\pi = 1$

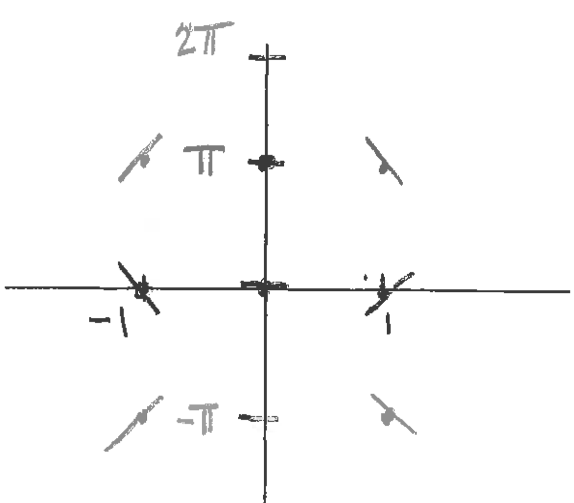
$y'(0,0) = 0$

$y'(1,-\pi) = -1$

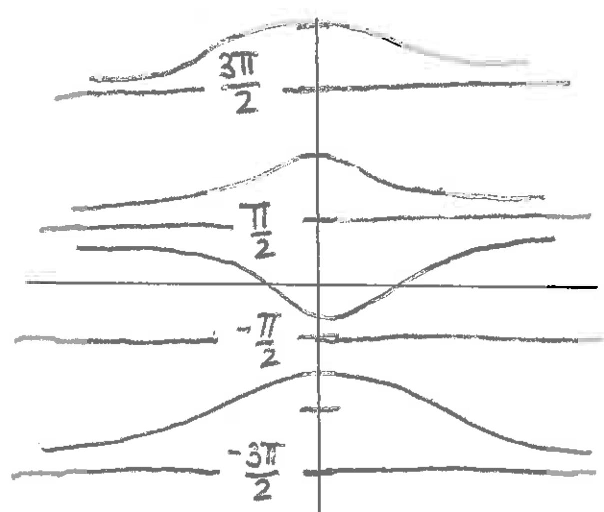
$y'(1,0) = 1$

$y'(1,\pi) = -1$

8 solns.



Vector Field



Phase Portrait

8. Solve $y''+4y'-5y=0; y(1)=0, y'(1)=1$. (24 points)

$$r^2+4r-5=0$$

$$(r+5)(r-1)=0$$

$$y = c_1 e^{-5x} + c_2 e^{x}$$

$$y' = -5c_1 e^{-5x} + c_2 e^{x}$$

$$y(1) = c_1 e^{-5} + c_2 e = 0$$

$$y'(1) = -5c_1 e^{-5} + c_2 e = 1$$

$$\left(\begin{array}{cc|c} e^{-5} & e & 0 \\ -5e^{-5} & e & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} e^{-5} & e & 0 \\ 0 & 6e & 1 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & e^{6} & 0 \\ 0 & 6e & 1 \end{array}\right)$$

$$c_2 = \frac{1}{6e} \qquad c_1 = -e^{6} c_2 = \frac{-e^{5}}{6}$$

$$y = \frac{-e^{5}}{6} e^{-5x} + \frac{1}{6e} e^{x}$$

9. Solve $y''+4y'+5y=0; y(0)=1, y'(0)=0$. (24 points)

$$r^2+4r+5=0$$

$$r = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$$

$$y' = -2c_1 e^{-2x} \cos x - c_1 e^{-2x} \sin x - 2c_2 e^{-2x} \sin x + c_2 e^{-2x} \cos x$$

$$y(0) = c_1 = 1$$

$$y'(0) = -2c_1 + c_2 = 0 \qquad c_2 = 2c_1 = 2$$

$$y = e^{-2x} \cos x + 2e^{-2x} \sin x$$