MATH 238 Sochacki Exam 3
June 6, 2014 Take Home Part 75 Points

Name KEY WEST
(Print) (1 point)

All necessary work must be shown for credit and must represent the question asked. Your work MUST be NEAT. You may NOT use computers. You may use your notes or text. All work must be your OWN work done by yourself.

I have neither received nor given help on this exam. Don Key
(Signature) (2 points)

1. Show that the IV ODE $y' = \sqrt[3]{1-y}; y(0) = 1$ has two solutions and explain why this is possible. (9 points)

$$\frac{dy}{dx} = (1-y)^{1/3} \qquad \text{EQ: } y = 1 \Rightarrow y(0) = 1$$

$$(1-y)^{-\frac{1}{3}}\,dy = dx$$

$$-\frac{3}{2}(1-y)^{\frac{2}{3}} = x + c$$

$$(1-y)^{\frac{2}{3}} = -\frac{2}{3}x + c$$

$$(1-y) = \left(-\frac{2}{3}x + c\right)^{\frac{3}{2}}$$

$$y = 1 - \left(-\frac{2}{3}x + c\right)^{3/2}$$

$$y(0) = 1 - c^{3/2} = 1$$

$$c = 0$$

$$y = 1 - \left(-\frac{2}{3}x\right)^{3/2}$$

$$f(x,y) = (1-y)^{1/3} \qquad \frac{\partial}{\partial y} f(x,y) = \frac{1}{3}(1-y)^{-\frac{2}{3}}(-1) = \frac{-1}{3(1-y)^{2/3}}$$

$$y \neq 1.$$

2. Solve $xy' - 2y = 2x^3 \sin 4x; y(1) = 1$. (9 points)

$$y' - \frac{2}{x}y = 2x^2 \sin 4x$$

$$p(x) = -\frac{2}{x} \qquad u(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = x^{-2}$$

$$\frac{d}{dx}\left(x^{-2}y\right) = x^{-2}y' - 2x^{-3}y = 2\sin 4x$$

$$x^{-2}y = -\frac{1}{2}\cos 4x + c$$

$$y = -\frac{1}{2}x^2\cos 4x + cx^2$$

$$y(1) = -\frac{1}{2}\cos 4 + c = 1 \Rightarrow c = 1 + \frac{1}{2}\cos 4$$

$$y = -\frac{1}{2}x^2\cos 4x + \left(1 + \frac{1}{2}\cos 4\right)x^2$$

3. Solve $y'' = 8xy' - 8x; y(0) = 1, y'(0) = 0$. (9 points)

$$y'' - 8xy' = -8x \qquad v = y'$$

$$v' - 8xv = -8x$$

$$p(x) = -8x \quad u(x) = e^{\int -8x\,dx} = e^{-4x^2}$$

$$\frac{d}{dx}\left(e^{-4x^2} v\right) = e^{-4x^2} v' - 8x e^{-4x^2} v = -8x e^{-4x^2}$$

$$e^{-4x^2} v = e^{-4x^2} + c_1$$

$$y' = v = 1 + c_1 e^{4x^2}$$

$$y'(0) = 1 + c_1 = 0 \quad c_1 = -1$$

$$y' = 1 - e^{4x^2}$$

$$\int_0^x y'(s)\,ds = \int_0^x (1 - e^{4s^2})\,ds$$

$$y(x) - y(0) = s\Big|_0^x - \int_0^x e^{4s^2}\,ds$$

$$y = 1 + x - \int_0^x e^{4s^2}\,ds$$

4. Solve $y''' - 3y' + 2y = 0$. (9 points)

$$r^3 - 3r + 2 = 0$$

$$r = 1$$

$$\begin{array}{r}
r^2 + r - 2 \\
r - 1 \;\big)\; r^3 + 0r^2 - 3r + 2 \\
r^3 - r^2 \\
\hline
r^2 - 3r + 2 \\
r^2 - r \\
\hline
-2r + 2 \\
-2r + 2
\end{array}$$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r = -2, \; r = 1$$

$$y = c_1 e^x + c_2 x e^x + c_3 e^{-2x}$$

5. Solve $y''+3y^2y'=0; y(0)=1, y'(0)=-1$. (9 points)

$v = y' \quad y'' = -3y^2y' = v'$

$\frac{dv}{dy} = \frac{v'}{y'} = \frac{-3y^2y'}{y'} = -3y^2$

$dv = -3y^2\,dy$

$v = -y^3 + c$

$y' = -y^3 + c$

$y'(0) = -1 = -y(0)^3 + c = -1 + c \Rightarrow c = 0$

$y' = -y^3$

$\frac{dy}{dx} = -y^3$

$y^{-3}dy = -dx$

$\frac{y^{-2}}{-2} = -x + c$

$y^{-2} = 2x + c$

$y = \frac{1}{(2x+c)^{\frac{1}{2}}}$

$y(0) = \frac{1}{\sqrt{c}} = 1$

$c = 1$

$y = \frac{1}{\sqrt{2x+1}}$

6. Solve $y''+4y = 2x^2-4x+3e^{-2x}$. (9 points)

$r^2 + 4 = 0$

$r = \pm 2i$

$y_H = c_1 \cos 2x + c_2 \sin 2x$

$y_p = Ax^2 + Bx + C + De^{-2x}$

$y_p' = 2Ax + B - 2De^{-2x}$

$y_p'' = 2A + 4De^{-2x}$

$y_p'' + 4y_p = 4Ax^2 + 4Bx + 4C + 2A + 8De^{-2x} = 2x^2 - 4x + 3e^{-2x}$

$A = \frac{1}{2} \quad B = -1 \quad 4C + 2A = 0 \quad D = \frac{3}{8}$

$C = -\frac{1}{2}A = -\frac{1}{4}$

$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}x^2 - x - \frac{1}{4} + \frac{3}{8}e^{-2x}$

7. Solve $x^2y''-2xy'+2y=\frac{1}{x}$ given that $y_1=x$ is a homogeneous solution. (18 points)

$$y_2 = uy_1$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

$$x^2y_2'' - 2xy_2' + 2y_2 = 0$$

$$x^2(u''y_1 + 2u'y_1' + uy_1'') - 2x(u'y_1 + uy_1') + 2uy_1 = 0$$

$$u\underbrace{[x^2y_1'' - 2xy' + 2y_1]}_{0} + u'[2x^2y_1' - 2xy_1] + u''x^2y_1 = 0$$

$$u'[2x^2\cdot 1 - 2x^2] + u''x^3 = 0$$

$$u''x^3 = 0$$

$$u'' = 0$$

$$u' = c$$

$$u = cx$$

$$y_2 = uy_1 = x\cdot x = x^2$$

$$W(y_1,y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

$$y_p = u_1y_1 + u_2y_2 = u_1x + u_2x^2$$

$$u_1 = \int \frac{\begin{vmatrix} 0 & x^2 \\ \frac{1}{x^3} & 2x \end{vmatrix}}{x^2}\,dx = \int -x^{-3}\,dx = \frac{x^{-2}}{2}$$

$$u_2 = \int \frac{\begin{vmatrix} x & 0 \\ 1 & \frac{1}{x^3} \end{vmatrix}}{x^2}\,dx = \int x^{-4}\,dx = \frac{x^{-3}}{-3}$$

$$y = c_1x + c_2x^2 + x\left(\frac{x^{-2}}{2}\right) + x^2\left(\frac{x^{-3}}{-3}\right)$$

$$y = c_1x + c_2x^2 + \frac{x^{-1}}{2} - \frac{x^{-1}}{3}$$

$$y = c_1x + c_2x^2 + \frac{x^{-1}}{6}$$

Bonus Point Problem (10 points): p. 228 No. 11.

$W = 8 = K\Delta L = K4 \Rightarrow K = 2$

$f = 1$

$mg = W \Rightarrow 32m = 8 \Rightarrow m = \frac{1}{4}$

$mu'' + fu' + Ku = \frac{1}{4}u'' + u' + 2u = 2\cos 2t$

$u'' + 4u' + 8u = 8\cos 2t$

$r^2 + 4r + 8 = 0$

$r = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm 2i$

$u_H = c_1 e^{-2t}\cos 2t + c_2 e^{-2t}\sin 2t$

$u_p = A\cos 2t + B\sin 2t$

$u_p' = -2A\sin 2t + 2B\cos 2t$

$u_p'' = -4A\cos 2t - 4B\sin 2t$

$u_p'' + 4u_p' + 8u_p = (4A + 8B)\cos 2t + (4B - 8A)\sin 2t$

$4A + 8B = 8$
$-8A + 4B = 0$ $\Rightarrow \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 1 & 2 & 2 \end{array}\right)$

$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 5 & 4 \end{array}\right)$

$B = \frac{4}{5}, A = \frac{2}{5}$

$u_p = \cos 2t + 2\sin 2t$

$u = c_1 e^{-2t}\cos 2t + c_2 e^{-2t}\sin 2t + \frac{2}{5}\cos 2t + \frac{4}{5}\sin 2t$

$u' = c_1(-2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t) + c_2(-2e^{-2t}\sin 2t + 2e^{-2t}\cos 2t)$

$-\frac{4}{5}\sin 2t + \frac{8}{5}\cos 2t$

$u'(0) = -2c_1 + 2c_2 + \frac{8}{5} = 0$

$u(0) = 0.5 = c_1 + \frac{2}{5}$

$c_1 = \frac{1}{10}$ $\qquad c_2 = -\frac{7}{10}$