MATH 248 Sochacki Exam 1
Oct., 3, 2017 100 Points

Name KEY WEST
(Print) (2 points)

All necessary work must be shown for credit and must represent the question asked. Your work MUST be NEAT. You may NOT use computers, notes or texts. Calculators can be used only to help with arithmetic.

I have neither received nor given help on this exam. Don Key
(Signature) (2 points)

1. Convert the following problems from the given base to the new base in the form asked. (24 points)

(a) 123 in base 10 to base 2

$$\begin{array}{r|l} 2 & 123 \\ 2 & 61 \quad 1 \\ 2 & 30 \quad 1 \\ 2 & 15 \quad 0 \end{array} \qquad \begin{array}{r|l} 2 & 7 \quad 1 \\ 2 & 3 \quad 1 \\ & 1 \quad 1 \end{array} \qquad (1111011)_2$$

(b) 1101101101101111 in base 2 to base 10

110 110 110 110 110 + 1

$2^{14}+2^{13}+2^{11}+2^{10}+2^{8}+2^{7}+2^{5}+2^{4}+2^{2}+2^{1}+1$

$2^{13}(3)+2^{10}(3)+2^{7}(3)+2^{4}(3)+2(3)+1$

$3(2^{13}+2^{10}+2^{7}+2^{4}+2)+1 = 6(2^{12}+2^{9}+2^{6}+2^{3}+1)+1$

$= 6\left(\frac{1-(2^3)^5}{1-2^3}\right)+1 = \frac{6}{7}(2^{15}-1)+1$

(c) $\frac{1}{7}$ in base 10 to base 2

$\frac{1}{7}$

$\frac{2}{7} \quad d_{-1}=0$

$\frac{4}{7} \quad d_{-2}=0$

$\frac{8}{7} \quad d_{-3}=1$

$\frac{1}{7} = (0.\overline{001})_2$

(d) $0.10\overline{110}$ in base 2 to base 10

0.10110110110...

$= 0.1 + 0.110110110 \times 2^{-2}$

$= \frac{1}{2} + \left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^7}+\frac{1}{2^8}+\cdots\right)\times 2^{-2}$

$= \frac{1}{2} + \left(\frac{3}{2^2}+\frac{3}{2^5}+\frac{3}{2^8}+\cdots\right)\times 2^{-2}$

$= \frac{1}{2} + \frac{3}{2^2}\left(1+\frac{1}{2^3}+\frac{1}{2^6}+\cdots\right)\times 2^{-2} = \frac{1}{2} + \left(\frac{3}{2^2}\,\frac{1}{1-\frac{1}{2^3}}\right)\frac{1}{2^2}$

$= \frac{1}{2} + \frac{3}{2^4-2} = \frac{1}{2}+\frac{3}{14} = \frac{10}{14} = \frac{5}{7}$

2. You have the 32 bits below to store a base 2 floating point number according to the standard rules covered in class. The first bit is for the sign and the last 9 bits are for the exponent.

									1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	3	3	3
1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2
																							8	7	6	5	4	3	2	1	0

exponent

Answer the following using our standard 32 bit guidelines. (24 points)

(a) Give the smallest positive number this 32 bit system can represent.

$0 \le exp \le 2^9 - 1 = 511$

$-256 \le exp \le 255$

smallest $(0.1)_2 \times 2^{-256} = 2^{-257}$

(b) Give the largest positive integer this 32 bit system can represent.

$0.\underbrace{111\ldots11}_{22} \times 2^{255}$ largest

$(1 - 2^{-22}) \times 2^{255} = 2^{255} - 2^{233}$

(c) How many numbers can this 32 bit system represent?

$2^{31} + 1 = 2{,}147{,}483{,}649$

(d) Give the the error and relative error in approximating 0.2 on this computer.

$32 - 9 - 1 = 22$ bits for mantissa

0.2

0.4 $d_{-1} = 0$

0.8 $d_{-2} = 0$

1.6 $d_{-3} = 1$

1.2 $d_{-4} = 1$

0.4 $d_{-5} = 0$

0.8 $d_{-6} = 0$

1.6 $d_{-7} = 1$

1.2 $d_{-8} = 1$

$0.2 = 0.0011\ 0011\ 0011\ldots$

$= 0.\underbrace{1100}_{4}\underbrace{1100}\underbrace{1100}\ldots \times 2^{-2}$

$5 \times 4 = 20 + 2$

$p = 0.\underbrace{1100}\underbrace{1100}\ldots\underbrace{1100}11 \times 2^{-2}$

$E = |0.2 - p| = |0.\underbrace{0\ldots0}_{22}\overline{0011} \times 2^{-2}|$

$= |0.\overline{0011} \times 2^{-24}| = |0.2 \times 2^{-24}|$

$R = 5|0.2 \times 2^{-24}| = 2^{-24}$

3. Let $p(x) = 6x^4 - 5x^3 + 4x^2 - 3x + 2$. (8 points)

(a) How many additions and how many multiplication are required to calculate $p(2)$ using this form of p?

4 additions

$1 + 2 + 3 + 4 = 10$ multiplications

(b) How many additions and how many multiplication are required to calculate $p(2)$ using Horner's algorithm?

4 adds
4 mults

$$p(x) = x(6x^3 - 5x^2 + 4x - 3) + 2$$
$$= x(x(6x^2 - 5x + 4) - 3) + 2 = x(x(x(6x - 5) + 4) - 3) + 2$$

(c) What is $p(2)$?

$$p(2) = \underbrace{2(\underbrace{2(\underbrace{2(\underbrace{6(2) - 5}_{7}) + 4}_{18}) - 3}_{33}) + 2}_{68}$$

4. Give the output of the following matlab code. (8 points)

```
m = 1/8+1/32;

i = 0;

while (rond(m) <= floor(m))

    m = m*2;
    i = i+1;

end

disp(m)
disp(i)

function n = rond(m)

    n = floor(m+0.5);

end
```

m	i
$\frac{5}{32} = 0.15625$	0
0.3125	1
0.625	2

0.625

2

5. Set up the bisection method to find the fixed point of cosine. How many iterations must be done to get the first five digits of this fixed point? (8 points)

$\cos x = x$

$\cos x - x = 0$

$g(x) = \cos x - x$

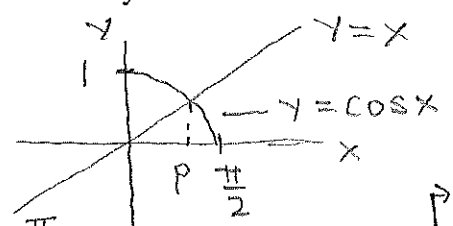
$g(0) = 1$

$g(\frac{\pi}{2}) = -\frac{\pi}{2}$

by IVT

$\cos x - x = 0$

$\cos x - x$



$x_1 = \frac{\frac{\pi}{2} + 0}{2} = \frac{\pi}{4}$

E: $\frac{\frac{\pi}{2} - 0}{2^n} \le 10^{-6}$

$\frac{\pi}{2} 10^6 \le 2^n$

$\ln \frac{\pi}{2} 10^6 \le n \ln 2$

$n \ge \frac{\ln \frac{\pi}{2} 10^6}{\ln 2}$

6. Show $f(x)=\frac{1}{x-1}$ has fixed points. What are the fixed points? If you use the FPM on f to try to find a fixed point with $x_1=1.5$, what are the next three terms in the algorithm under f? Explain what is happening. (12 points)

$$f(x)=\frac{1}{x-1}=x$$

$$1=x(x-1)$$

$$1=x^2-x$$

$$x^2-x-1=0$$

$$x=\frac{1\pm\sqrt{1+4}}{2}$$

$$x=\frac{1\pm\sqrt{5}}{2}$$

$$\approx 1.618,\ 0.618$$

$$x_1=1.5$$

$$f(1.5)=2$$

$$f(2)=1$$

$$f(1)=\text{DNE}$$

$$f'(x)=\frac{-1}{(x-1)^2}$$

$$|f'(x)|=\frac{1}{(x-1)^2}$$

$$|f'(1.5)|=\frac{1}{(0.5)^2}=4>1$$

7. You are to approximate $p=\sqrt{3}$ as follows. (12 points)

(a) Give the function f for a Newton's Method to determine p.

$$x=\sqrt{3}$$

$$x^2=3$$

$$g(x)=x^2-3$$

$$f(x)=x-\frac{g(x)}{g'(x)}=x-\frac{x^2-3}{2x}$$

$$=\frac{1}{2}x+\frac{3}{2x}$$

(b) If $x_1=2$ what are the next three iterates given by your Newton's Method?

$$f(2)=1+\frac{3}{4}=\frac{7}{4}$$

$$f\left(\frac{7}{4}\right)=\frac{7}{8}+\frac{3}{\frac{7}{2}}=\frac{7}{8}+\frac{6}{7}=\frac{97}{56}$$

$$f\left(\frac{97}{56}\right)=\frac{97}{112}+\frac{3}{\frac{97}{28}}$$

$$=\frac{97}{112}+\frac{84}{97}$$

(c) What is $f'(p)$?

$$f'(p)=0$$

$$f'(x)=\frac{1}{2}-\frac{3}{2x^2}$$

$$f'(\sqrt{3})=\frac{1}{2}-\frac{1}{2}=0$$

(d) What is $f''(p)$?

$$f''(x)=\frac{3}{x^3}$$

$$f''(\sqrt{3})=\frac{3}{(\sqrt{3})^3}=\frac{1}{\sqrt[3]{3}}$$